The Eulerian- and Lagrangian-Mean Meridional Circulations in the Stratosphere at the Time of a Sudden Warming

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ABSTRACT

The Eulerian-mean (usual zonal mean) and the Lagrangian-mean (an average in the zonal direction along a curved "material line," consisting of definite individual air particles) meridional circulations are investigated, given an upward propagating planetary wave incident on a critical level. The Eulerian-mean circulation is such that an upward motion appears at higher latitudes, a downward motion at lower latitudes, and an equatorward meridional flow near the critical level. The Lagrangian circulation is quite different from the Eulerian one. A strong poleward meridional flow appears concentrated at the critical level. In connection with this meridional flow, a Lagrangian-mean vertical motion occurs which diverges from this level at higher latitudes, i.e., it consists of a downward motion below the critical level and an upward motion above the level. At lower latitudes, vertical motions converge toward the critical level. The vertical motion field of this four-sector structure in the meridional section causes corresponding zonal mean temperature changes as observed in real sudden warmings.

From a set of equations derived by Andrews and McIntyre (1978b), which govern the Lagrangian-mean quantities, we can obtain the above results directly. By interpreting the physical meaning of this system of equations, we interpret the strong meridional flow at the critical level as an Ekman-transport-like motion caused by a strong westward force at the critical level, which arises from a sharp gradient of the zonal component of the radiation stress associated with the wave. The mechanism of a sudden warming as viewed in terms of the Lagrangian-mean motion is discussed.

1. Introduction

The mean meridional circulation of the stratosphere in wintertime has the feature of a two-cell structure (e.g., Miyakoda, 1963). One of the two cells located at high latitudes is an indirect cell with upward motion in the polar region and downward motion in the middle latitude region. The magnitude of these vertical motions can be 3 mm s^{-1} or more (Vincent, 1968), so that the vertical displacement of the air due to these motions should amount to more than 10 km in a month. Hence, the motions may exert a great influence on the distributions of ozone and other trace substances and on the potential temperature in the stratosphere. In reality, however, there is another transport mechanism, i.e., the eddy transport by planetary waves. This effect is equally large, tends to oppose the transport by the mean circulation, and in budget calculations of those quantities the contributions from the two effects almost cancel each other (Manabe and Hunt, 1968; Hunt and Manabe, 1968; Mahlman, 1969; Mahlman and Moxim, 1978). The cancellation of the two effects is not accidental because the mean circulation includes a component which is forced by the eddy heat transport by planetary waves. Both the eddy transport effect and the induced mean circulation are second-order effects of the wave and there is a definite relationship between them.

Concerning the second-order effects of planetary waves, Charney and Drazin (1961) showed that the waves do not cause changes of zonal mean fields, provided that the waves are steady (or propagating in the zonal direction with a constant phase velocity) and provided that there are no critical surfaces (where zonal wind velocity equals the wave's phase velocity) nor dissipative effects. This result is often referred to as the Charney–Drazin (CD) theorem. Extension and generalizations of the theorem, and also unification of it with a similar theorem by Eliassen and Palm (1961), have recently been made by Andrews and McIntyre (1976, 1978a,b,c), Boyd (1976) and Bretherton (1979). (Also see Holton, 1974.) One of the present authors

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(Nakamura, 1979) discussed the same problem considering energy and momentum balances of a material part of a fluid and obtained similar results.

Although Charney and Drazin did not discuss motions of individual air particles explicitly, the implication of the CD theorem for this problem is rather clear. Since the theorem states that zonal mean temperature does not change with time despite the presence of stable stratification and adiabatic condition, it implies that individual air particles do not make permanent displacements in the vertical, even though they make temporary excursions by the effect of the waves. In short, the Lagrangian-mean vertical velocity is zero when the CD theorem holds. This fact may seem contradictory to the existence of the mean vertical motions mentioned earlier. However, two different procedures of taking a mean are in question. Following the Riehl and Fultz idea (1957), Mahlman (1969) considered vertical motions averaged zonally referring to meandering jet stream paths, instead of averaged along a constant latitude circle. He showed that at the time of a sudden warming event the vertical motions thus calculated are downward on the poleward side and upward on the equatorward side of the polar jet, even though the mean vertical motions in the usual sense are directed opposite. The necessity of such a concept for understanding tracer transport problems was expressed by Züllig (1973). The Lagrangian mean discussed in the present study is similar to the average considered by Mahlman (1969), though not identical in exact sense. Concerning time-varying ocean currents, Longuet-Higgins (1969) has pointed out that the Eulerian-mean current and the Lagrangian-mean current differ from each other the difference, in general, being defined as the Stokes drift velocity. An analogous situation exists in the present problem (Uryu, 1974a). The difference between the two circulations is demonstrated clearly in the numerical experiments conducted by Kida (1977).

The CD theorem breaks down when any of the conditions mentioned previously is violated. The Lagrangian-mean vertical (and meridional) velocity may no longer then be zero. In particular, in the sudden warming situation drastic changes of zonal mean temperature and velocity fields are caused by the second-order effects of planetary waves (Matsuno, 1971). It is of interest to investigate the Lagrangian- as well as the Eulerian-mean meridional circulations at the time of a sudden warming. In the present study, we discuss the two circulations by considering a situation where an upward-propagating planetary wave is incident on a critical level and is causing changes of the zonal mean fields. The problem is similar to the one treated by Matsuno (1971) and also by Clark (1974). We also discuss the problem in connection with the system of equations for Lagrangian-means recently derived by Andrews and McIntyre (1978b) and we present a new view on the mechanism of the sudden warming.

2. The model and basic equations

The problem we treat in the present paper is the same as that considered in Section 2 of Matsuno (1971). Namely, we consider an adiabatic and frictionless atmosphere on a beta plane, bounded laterally at two latitudes but not bounded in the zonal or vertical direction. As the basic state, we assume a zonal flow which is uniform horizontally but variable with height. The zonal flow is westerly below a level z_c and easterly above the level. Thus, z_c is a critical level for stationary waves. We assume that there is a stationary wave source far below the level z_c, so that a steadily upward-propagating planetary wave is set up in this model atmosphere. Here we consider a dissipation-dominated (hence absorbing) critical layer rather than a nonlinear critical layer discussed in connection with lateral propagation of planetary waves (e.g., Béland, 1976; Tung, 1979).

The governing equations for the stationary planetary wave and the changes of zonal mean fields forced by second-order effects of the wave can be written with the hydrostatic and the geostrophic approximations as follows (cf. Holton, 1975).

Disturbance potential vorticity equation

$$\bar{u}(z)\frac{\partial}{\partial x}\left[\nabla^2\phi+\frac{f^2}{N^2}\left(\frac{\partial^2\phi}{\partial z^2}-\frac{1}{H}\frac{\partial\phi}{\partial z}\right)\right]+\bar{q}_y\frac{\partial\phi}{\partial x}=0. \quad (2.1)$$

Mean field equations

$$\frac{\partial\bar{u}}{\partial t}-f\bar{v}=-\frac{1}{f^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}}\right) \quad (2.2)$$

$$f\bar{u}=-\frac{\partial\bar{\phi}}{\partial y} \quad (2.3)$$

$$\frac{\partial\bar{v}}{\partial y}+\frac{\partial\bar{w}}{\partial z}-\frac{1}{H}\bar{w}=0 \quad (2.4)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\bar{\phi}}{\partial z}\right)+N^2\bar{w}=-\frac{1}{f}\frac{\partial}{\partial y}\left(\overline{\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}}\right). \quad (2.5)$$

Here the following notation is used:

- x, y eastward and northward coordinates
- z vertical log-pressure coordinate, with p_0 as reference pressure $[=-H\ln(p/p_0)]$
- H scale height of reference atmosphere
- f Coriolis parameter (constant)
- N Brunt-Väisälä frequency (constant)
- ϕ zonally variable part of the perturbation of geopotential height
- $\bar{\phi}$ zonal mean geopotential height
- $\bar{u}$, $\bar{v}$, $\bar{w}$ zonal mean velocity components in the (x,y,z) directions
- $\bar{q}_y$ latitudinal gradient of zonal mean potential vorticity.

Eliminating $\bar{u}$, $\bar{v}$ and $\bar{w}$ in (2.2)–(2.5), we obtain the zonal mean potential vorticity equation for a single variable $\bar{\phi}$ as

$$\left[\frac{\partial^2}{\partial y^2}+\frac{f^2}{N^2}\left(\frac{\partial^2}{\partial z^2}-\frac{1}{H}\frac{\partial}{\partial z}\right)\right]\frac{\partial\bar{\phi}}{\partial t}=-\frac{1}{f}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\overline{\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}}\right)+\frac{f^2}{N^2}\left(\frac{\partial}{\partial z}-\frac{1}{H}\right)\left(\overline{\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}}\right)\right], \quad (2.6)$$

where the overbar expresses averaging in the x direction.

The boundary condition in the y direction is

$$\left.\begin{aligned} v=\frac{1}{f}\frac{\partial\phi}{\partial x}=0 \\ \bar{v}=0 \end{aligned}\right\} \quad \text{at} \quad y=0, D, \quad (2.7a)$$

where D is the lateral separation of the two bounding walls.

In the following analysis, we regard $\bar{u}$ in (2.1) as constant in time and denote it by u_0. Since $\partial\bar{u}/\partial t$ caused by the wave is second order in wave amplitude, the resultant error of wave structure is at most third order and the whole procedure may be justified. In what follows, we first note a certain property of solutions of (2.1) and then obtain a solution of (2.6) by evaluating the right-hand side by making use of that property. Once we have obtained $\partial\bar{\phi}/\partial t$, we can readily determine $\bar{v}$ and $\bar{w}$ from (2.2)–(2.5), so that we get the mass streamfunction of Eulerian-mean meridional circulation χ. The Lagrangian-mean meridional and vertical velocities will be determined by adding the Stokes velocity corrections to respective components of the Eulerian-mean circulation.

3. Eulerian-mean fields

a. *Disturbance and eddy heat flux*

As a solution of (2.1), we consider the form

$$\phi=\varphi(z)\sin\frac{\pi}{D}ye^{ikx}, \quad (3.1)$$

which satisfies the boundary condition (2.7a). Further, by separating the density weight factor as

$$\varphi(z)=e^{z/2H}\psi(z),$$

we obtain the following equation for ψ in canonical form:

$$\frac{d^2\psi}{dz^2}+\left\{\left[\frac{\bar{q}_y}{u_0}-\left(k^2+\frac{\pi^2}{D^2}\right)\right]\frac{N^2}{f^2}-\frac{1}{4H^2}\right\}\psi=0. \quad (3.2)$$

Solutions of (3.2) for several $\bar{u}(z)$ profiles were discussed by Charney and Drazin (1961). For the present purpose, however, it is not necessary to obtain explicit forms of the solutions, but it is sufficient only to evaluate the quantity entering into the second term[3] on the right-hand side of (2.6). The quantity is related to eddy heat flux, and its behavior has been extensively discussed in literature on the wave propagation problem as well as in other contexts (e.g., Charney and Drazin, 1961; Kuo, 1949). Following these discussions, we derive from (3.2)

$$\mathrm{Im}\left(\psi^*\frac{d\psi}{dz}\right)=\frac{N^2}{f^2}\pi\left(\frac{\bar{q}_y}{|\partial u_0/\partial z|}\right)_{z_c}|\psi(z_c)|^2\mathsf{H}(z_c-z), \quad (3.3)$$

where z_c is the critical level, i.e., $\bar{u}(z_c)=0$, and H is the Heaviside step function, defined as

$$\mathsf{H}(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0.\end{cases}$$

Here we have assumed that the wave is upward propagating.

Then considering the functional form of ϕ given by (3.1), we can express the right-hand side of (2.6) as

$$\frac{1}{f}\left(\overline{\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial x}}\right)=\frac{k}{2f}\mathrm{Im}\left(\varphi^*\frac{d\varphi}{dz}\right)\sin^2\frac{\pi}{D}y$$

$$=\frac{N}{f}Q_0\sin^2\frac{\pi}{D}ye^{(z-z_c)/H}\mathsf{H}(z_c-z), \quad (3.4)$$

where

$$Q_0=\frac{\pi Nk}{2f^2}\left(\frac{\bar{q}_y}{|\partial u_0/\partial z|}|\varphi|^2\right)_{z_c}. \quad (3.5)$$

b. *Solution for* $\partial\bar{\phi}/\partial t$

Inserting (3.4) into (2.6) and rewriting the resultant equation with the use of new dimensionless coordinates $\tilde{y}$ and $\tilde{z}$ defined as

$$\tilde{y}=(\pi/D)y, \quad \tilde{z}=\frac{N}{f}\frac{\pi}{D}(z-z_c), \quad (3.6)$$

we obtain

$$\left(\frac{\partial^2}{\partial\tilde{y}^2}+\frac{\partial^2}{\partial\tilde{z}^2}-\frac{1}{\tilde{H}}\frac{\partial}{\partial\tilde{z}}\right)\frac{\partial\bar{\phi}}{\partial t}=Q_0\sin 2\tilde{y}\cdot\delta(\tilde{z}), \quad (3.7)$$

where

$$\tilde{H}=\frac{N}{f}\frac{\pi}{D}H.$$

Next, we consider the lateral boundary condition to solve (3.7). Since the momentum flux vanishes in the

[3] The first term on the right of (2.6) vanishes for solutions of the form (3.1).

present case, we have from (2.1) and (2.7b)

$$\frac{\partial \bar{u}}{\partial t}=0 \quad \text{at} \quad y=0, D.$$

This condition is rewritten using (2.3) as

$$\frac{\partial}{\partial \tilde{y}}\left(\frac{\partial \bar{\phi}}{\partial t}\right)=0 \quad \text{at} \quad \tilde{y}=0, \pi. \tag{3.8}$$

Because of this condition, it is impossible to get solutions by assuming the same lateral structure as the forcing term, though Clark (1974) did so. To obtain a solution satisfying the boundary condition, we expand $\partial\bar{\phi}/\partial t$ as

$$\frac{\partial \bar{\phi}}{\partial t}=\sum_{m=\text{odd}} G_m(\tilde{z}) \cos m\tilde{y}, \tag{3.9}$$

where m takes odd integer values starting from 1. We also expand the inhomogenous term in the same way (cf. Uryu, 1974b):

$$\sin 2\tilde{y}=\sum_{m=\text{odd}}\left(-\frac{1}{\pi}\frac{8}{m^2-4}\right)\cos m\tilde{y}. \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.7) and equating each component, we now obtain equations for the G's:

$$\frac{d^2G_m}{d\tilde{z}^2}-\frac{1}{\tilde{H}}\frac{dG_m}{d\tilde{z}}-m^2G_m=-Q_0\frac{1}{\pi}\frac{8}{m^2-4}\delta(\tilde{z}). \tag{3.11}$$

The solution of (3.11) which satisfies the boundary condition that G_m should not grow in either direction away from $\tilde{z}=0$ is readily obtained as

$$G_m(\tilde{z})=\begin{cases} C_m \exp(\mu_m^- \tilde{z}), & \tilde{z}>0 \\ C_m \exp(\mu_m^+ \tilde{z}), & \tilde{z}<0 \end{cases} \tag{3.12}$$

where

$$C_m=Q_0\frac{1}{\pi\mu_m}\frac{4}{m^2-4}$$

$$\mu_m^{\pm}=\frac{1}{2\tilde{H}}\pm\mu_m,$$

$$\mu_m=\left(\frac{1}{4\tilde{H}^2}+m^2\right)^{\frac{1}{2}}.$$

This completes the solution of (3.7). The convergence of the series $\sum G_m \cos m\tilde{y}$ is fairly rapid because $G_m \propto m^{-3}$ for a large m. An approximate picture of $\partial\bar{\phi}/\partial t$ can be gained by considering the first term. Since C_1 is negative, we note that $\partial\bar{\phi}/\partial t \gtrless 0$ according as $\tilde{y} \gtrless \pi/2$, namely, zonal mean geopotential height rises at higher latitudes and falls at lower latitudes.

c. Other Eulerian-mean quantities

The acceleration of zonal mean flow $\partial\bar{u}/\partial t$ is obtained from $\partial\bar{\phi}/\partial t$ simply by the geostrophic wind relation, *viz.*,

$$\frac{\partial \bar{u}}{\partial t}=\frac{\pi}{fD}Q_0\sum mG_m(\tilde{z}) \sin m\tilde{y}. \tag{3.13}$$

The resulting series is still absolutely and uniformly convergent, and term-by-term differentiation is justified. To compute quantitatively the zonal acceleration, we adopt the following values: $N=2\times10^{-2}$ s^{-1}, $f=10^{-4}$ s^{-1}, $H=7$ km, $D=5000$ km, $k=2\pi/(10\,000$ km), $\bar{q}_y=\beta(60°)=1.1\times10^{-11}$ s^{-1} m^{-1} $|\partial u_0/\partial z|_{z_c}=3\times10^{-3}$ s^{-1}, $|\varphi(z_c)|=5000$ m^2 s^{-2}. These parameter values are the same as those adopted by Matsuno (1971) for a similar calculation, the results of which are shown in Fig. 1 in that paper, except that k in the present calculation is twice as large as the former value. The present value corresponds to zonal wavenumber 2 along the 60° latitude circle. Consequently, results turn out to be larger by a factor of 2 compared with the corresponding former ones. Since the critical level in the real situation is not well-defined, those values at that level have large uncertainties. An alternative way is to consider the upward energy flux associated with the wave. According to Eliassen and Palm (1961), the wave energy flux is given (within the geostrophic approximation) by

$$F(y,z)=\frac{\bar{p}}{N^2}u_0(z)\overline{\left(\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\right)}, \tag{3.14}$$

where $\bar{p}(z)$ is the basic-state air density. Combining (3.14) with (3.4), we can get the energy flux implicitly assumed in our calculation. Considering a level located 15 km below the critical level and that $\bar{u}_0$ there is 10 m s^{-1} (we are considering the 100 mb level), we obtain the maximum of the energy flux at that level to be ~800 ergs cm^{-2} s^{-1}. This value seems too large, by a factor of about 2 or less, when compared with the corresponding observational values (Julian and Labitzke, 1965; Miller and Johnson, 1970). Therefore, the numberical results presented here are considered to be larger, by a factor of about 2, than the real situation. The result of calculation of $\partial\bar{u}/\partial t$ is shown in Fig. 1. The zonal mean meridional velocity $\bar{v}$ is then easily obtained as

$$\bar{v}=-\frac{1}{f}\frac{\partial\bar{u}}{\partial t}.$$

The zonal mean temperature change is given as

$$\frac{\partial \bar{T}}{\partial t}=\frac{H}{R}\frac{\partial}{\partial t}\left(\frac{\partial\bar{\phi}}{\partial z}\right)$$

$$=\frac{H}{R}\frac{N\pi}{fD}Q_0\sum\frac{dG_m}{d\tilde{z}}\cos m\tilde{y}, \tag{3.15}$$

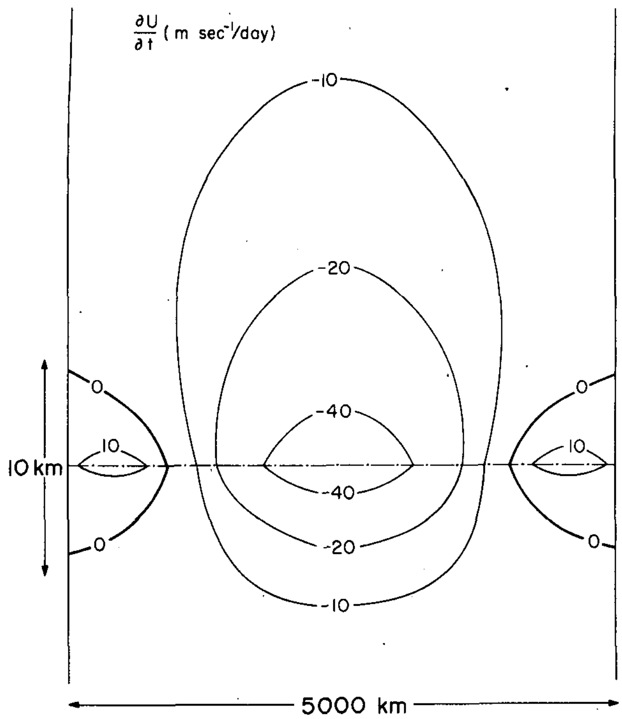

FIG. 1. Meridional cross section of the zonal mean acceleration (m s^{-1} day^{-1}) caused by planetary waves incident on a critical level (shown by a chain line).

where

$$\frac{dG_m}{d\tilde{z}}=\begin{cases}\mu_m^- C_m \exp(\mu_m^-\tilde{z}), & \tilde{z}>0\\ \mu_m^+ C_m \exp(\mu_m^+\tilde{z}), & \tilde{z}<0\\ \text{indefinite}, & \tilde{z}=0.\end{cases}\tag{3.16}$$

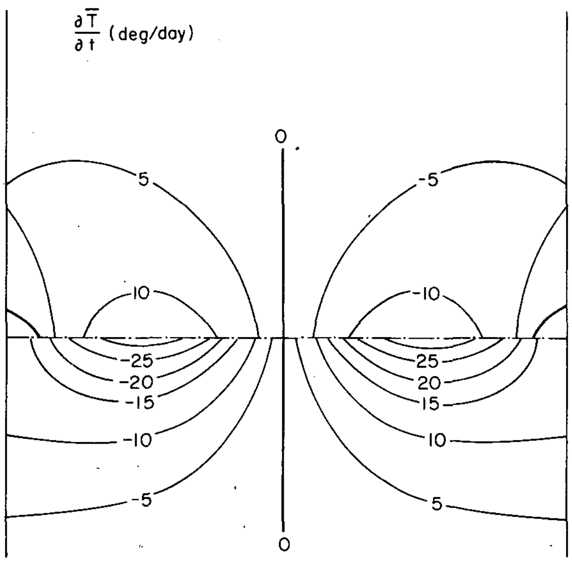

FIG. 2. As in Fig. 1 except for the zonal mean temperature change (K day^{-1}) .The temperature change has a discontinuity at the critical level. The poleward side is on the right.

Apparently, the temperature tendency has a discontinuity at the critical level, arising from the discontinuity of the eddy heat flux. The distribution of $\partial\bar{T}/\partial t$ in the meridional plane thus calculated is shown in Fig. 2. The results shown in both Figs. 1 and 2 are the same as Fig. 1 of Matsuno (1971), except for a numerical factor, although details of the mathematical analysis were not presented there. The zonal mean vertical velocity $\bar{w}$ is obtained from Eq. (2.5). Inserting $\partial/\partial t(\partial\bar{\phi}/\partial z)$ obtained from (3.9) and the eddy heat flux (3.4) into (2.5), we have

$$\bar{w}=-\frac{\pi Q_0}{fND}\left[e^{\tilde{z}/\tilde{H}}\sin 2\tilde{y}\mathbf{H}(-\tilde{z})+\sum\frac{1}{\pi\mu_m}\frac{4}{m^2-4}\frac{dG_m}{d\tilde{z}}\cos m\tilde{y}\right].\tag{3.17}$$

As seen above, both the first and second terms have a discontinuity at $\tilde{z}=0$, but the sum has no discontinuity. We now obtain the mass streamfunction χ of the Eulerian-mean meridional circulation, which is defined as

$$\left.\begin{aligned}\bar{p}(z)\bar{v}&=-\frac{\partial\chi}{\partial z}\\ \bar{p}(z)\bar{w}&=\frac{\partial\chi}{\partial y}\end{aligned}\right\}.\tag{3.18}$$

Integrating $\bar{w}$ with respect to $\tilde{y}$ under a suitable boundary condition, we have

$$\chi=-\bar{p}_c\frac{Q_0}{fN}\left[\sin^2\tilde{y}\mathbf{H}(-\tilde{z})+e^{\tilde{z}/\tilde{H}}\sum\frac{1}{\pi\mu_m}\frac{4}{m(m^2-4)}\frac{dG_m}{d\tilde{z}}\sin m\tilde{y}\right].\tag{3.19}$$

The results of numerical computation of χ and $\bar{w}$ with the given parameter values are shown in Figs. 3 and 4, respectively. As seen from Fig. 3 there is a major circulation cell (not closed) with an upward motion on the poleward side and a downward motion on the lower latitude side. The gross feature of the circulation has already been suggested by Matsuno (1971) based on qualitative considerations. The vertical velocity reaches about 2 cm s^{-1} at its maxima. As remarked previously, this value may be too large (by a factor of 2) compared with the real situation.

In addition to the major cell, two small counter cells appear in the vicinity of bounding walls. The origin of these cells may be explained in the following way. Because of the boundary condition (2.8), we have

$$\frac{\partial^2}{\partial y\partial t}\left(\frac{\partial\bar{\phi}}{\partial z}\right)=0,$$

implying that the zonal mean temperature change must be almost constant near the boundaries. On the other hand, the convergence of eddy heat flux becomes zero at the boundary (the eddy flux itself has a double zero), as pointed out by Holton (1976), so that $\bar{w}$ at a boundary should occur in such a way that it causes the same temperature tendency as does the eddy effect in the vicinity of the boundary. This is in contrast to the situation in most of the channel where the two effects are counteracting. The mechanism is similar to that of the formation of three cells in a dry baroclinic atmosphere on a bounded β-plane (Phillips, 1954). As discussed and evidenced in a numerical model by Holton (1976), if disturbance of zonal wavenumber 1 on a sphere is included, upward motion dominates in the entire polar region as in the real stratosphere. Thus, the appearance of the two small counter cells is merely a reflection of the artificial geometrical constraint and should not be given particular significance. Corresponding to these circulations, westerly acceleration appears near the walls (Fig. 1), opposing the general easterly acceleration. In the numerical experiments of the stratospheric sudden warming by Matsuno (1971) and by Holton (1976), westerly acceleration appeared in the polar region at earlier stages of the evolution in the case of wavenumber 2 forcing. It is possible that some part of the acceleration is caused in the same way as discussed above. If this is the case, coexistence of the wavenumber 1 component may have considerable influence upon the evolution of zonal flow in the polar region because the easterly acceleration in the polar region by wavenumber 1 may help the penetration of succeeding waves.

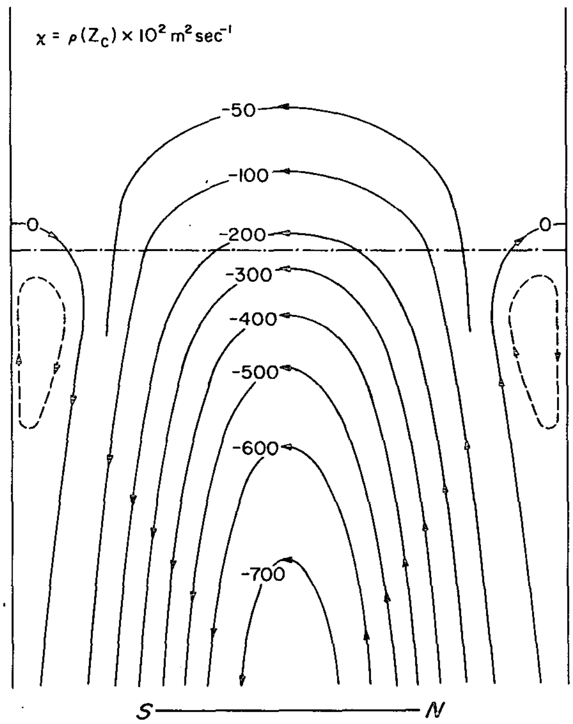

FIG. 3. The mass streamfunction for the Eulerian-mean meridional circulation $\bar{\rho}(z_c)\times 10^2$ m² s⁻¹, where $\bar{\rho}(z_c)$ is the density at the critical level.

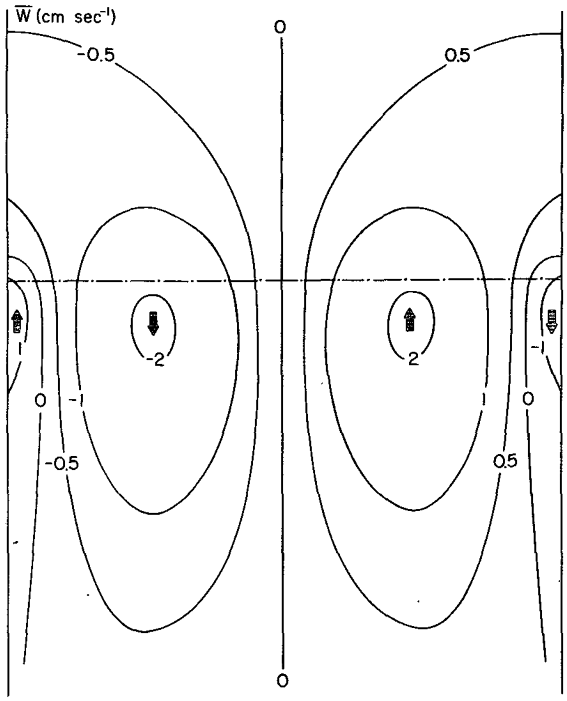

FIG. 4. Meridional cross section of the Eulerain-mean vertical velocity (cm s⁻¹).

4. Lagrangian-mean meridional circulation

This section concerns movements of individual air particles in the presence of wave disturbances. In particular, we consider the Lagrangian-mean motion, the mean motion of a set of air particles that forms a thin continuous material tube nearly along a latitude circle. More specifically, we take a tube that would lie at a constant latitude and at a constant height (therefore, a straight line in the zonal direction in our β-plane geometry) if the wave were not present. In this case, each member particle of the tube has identical motion, except for its initial phase, under the influence of the wave. Thus, the Lagrangian-mean motion following a single particle for a period longer than the wave period is the same for all member particles, and it coincides with the instantaneous value of velocity averaged over a sufficient number of member particles. In short, the Lagrangian-mean motion is the motion of the center of mass of the wavy material tube. This is a special case of the slightly more general "generalized Lagrangian-mean" concept recently introduced by Andrews and McIntyre (1978b). By its definition, the Lagrangian-mean motion is suitable for the understanding of movements of trace substances (Dunkerton, 1978).

The Lagrangian-means of any variables can be obtained based on the distribution of the variables described in the Eulerian coordinates, provided that the wave amplitude is small (Bretherton, 1971). Andrews and McIntyre (1978b) have recently developed a theory to describe the Lagrangian-means quite generally, including finite amplitude waves. The Lagrangian-mean of a variable B is given as

$$\bar{B}^{\mathrm{L}}(y,z)=\int B(x+\xi,\, y+\eta,\, z+\zeta)dx \Big/ \int dx, \quad (4.1)$$

where ξ, η and ζ are displacements of air particles from their unperturbed positions due to the wave perturbation, and are functions of $\mathbf{x}$ and t in the Eulerian sense. The displacement $\boldsymbol{\xi}$ is connected with the Eulerian velocity field $\mathbf{V}$ by the kinematical relation

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_0\cdot\nabla\right)\boldsymbol{\xi}=\mathbf{V}(\mathbf{x}+\boldsymbol{\xi})-\mathbf{V}_0(\mathbf{x}), \quad (4.2)$$

where $\mathbf{V}_0$ is the unperturbed velocity field and the velocity $\mathbf{V}$ should be evaluated at the position $\mathbf{x}+\boldsymbol{\xi}$, while $\mathbf{V}_0$ and $\boldsymbol{\xi}$ are those at $\mathbf{x}$. The definition of $\boldsymbol{\xi}$ here is slightly different from that adopted by Andrews and McIntyre (1978b; see also Bretherton, 1971). Their ξ is defined as the particle's displacement from its reference position which is moving with the Lagrangian-mean velocity, the velocity which includes not only the basic-state velocity V_0 but also the mean velocity associated with the wave itself. Then the definitions of $\boldsymbol{\xi}$ and the Lagrangian-mean velocity become mutually dependent and uneasy for understanding, though this way is more suited to rigorous treatments of finite-amplitude waves. Here we take V_0 as the reference flow, so that the definition of $\boldsymbol{\xi}$ become one way and explicit, as shown by (4.2). Thus we take a naive and easier way at the expense of generality and rigorousness. The results are the same between the two methods up to the second order of wave amplitude.

Now, assuming that the Eulerian perturbation velocity $\mathbf{v}(=\mathbf{V}-\mathbf{V}_0)$ and the displacement $\boldsymbol{\xi}$ are small and that $b(=B-B_0)$, the Eulerian deviation of the variable B from its basic-state value B_0 is also small, we get approximate equations for (4.1) and (4.2) as follows:

$$\bar{B}^{\mathrm{L}}=\int \{B_0+b+\boldsymbol{\xi}\cdot\nabla B_0+\boldsymbol{\xi}\cdot\nabla b + \tfrac{1}{2}[\boldsymbol{\xi}\boldsymbol{\xi}]:[\nabla\nabla]B_0\}_{\mathbf{x}}dx \Big/ \int dx, \quad (4.3)$$

$$\left[\frac{\partial}{\partial t}+\mathbf{V}_0(\mathbf{x})\cdot\nabla\right]\boldsymbol{\xi}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t)+(\boldsymbol{\xi}\cdot\nabla\mathbf{V}_0)_{\mathbf{x}}. \quad (4.4)$$

In the above and subsequent equations, a symbol such as $[\mathbf{pq}]$ is used to represent a dyadic tensor composed of vectors $\mathbf{p}$ and $\mathbf{q}$ and the double dot-product of these two tensors stands for contraction with respect to two indices, i.e.,

$$[S]:[T]=\sum_i\sum_j S_{ij}T_{ij}.$$

The last term of the integrand of (4.3) represents the second-order term of Taylor expansion of $B(\mathbf{x}+\boldsymbol{\xi})$ about $\mathbf{x}$. By using the above equations, we can determine $\bar{B}^{\mathrm{L}}$ and $\boldsymbol{\xi}$ because b and $\mathbf{v}$ are known functions as solutions of the perturbation equations. Note that $\bar{B}^{\mathrm{L}}$ given by (4.3) retains terms which are second order in wave amplitude. For consistency, $\boldsymbol{\xi}$ in the first term in the bracket, $\boldsymbol{\xi}\cdot\nabla B_0$, should be correct to the second order, which is not given by (4.4). Fortunately, in the present problem the term is not important for evaluation of the Lagrangian-mean meridional or vertical velocities, because B_0 in this case (the basic state meridional or vertical velocity) is zero.[4] For the same reason, the last term vanishes for $B=v$ or w, so that we have

$$\left.\begin{aligned}\bar{v}^{\mathrm{L}}&=\bar{v}+\overline{\boldsymbol{\xi}\cdot\nabla v}\\ \bar{w}^{\mathrm{L}}&=\bar{w}+\overline{\boldsymbol{\xi}\cdot\nabla w}\end{aligned}\right\}, \quad (4.5)$$

where the overbars stand for the usual zonal mean operation. From (4.5), we understand that the Lagrangian-mean velocities differ from the corresponding Eulerian ones by the amounts given as second terms, and which are often referred to as the Stokes drift or as the Stokes velocity (Longuet-Higgins, 1969; Bretherton, 1971; Wallace, 1978).

Note that the expression of Stokes drift as appearing in (4.5) is not correct for sheared flows (in the zeroth order) as pointed out by Andrews and McIntyre (1978b). In the present problem, for example, the zonal component of Stokes drift should have an additional term $\frac{1}{2}\overline{\zeta^2}\partial^2u_0/\partial z^2$.

Since we have already obtained the Eulerian-mean meridional circulation, which is caused by the planetary wave as a second-order effect, we can get the Lagrangian-mean circulation by evaluating the Stokes velocity.

In the present case, (4.4) is written as

$$\left.\begin{aligned}\bar{u}_0\frac{\partial\xi}{\partial x}&=u+\zeta\frac{\partial\bar{u}_0}{\partial z}\\ \bar{u}_0\frac{\partial\eta}{\partial x}&=v\\ \bar{u}_0\frac{\partial\zeta}{\partial x}&=w\end{aligned}\right\}. \quad (4.6)$$

[4] Andrews and McIntyre (1978b) deal with this problem by defining $\boldsymbol{\xi}$ slightly differently so that $\bar{\boldsymbol{\xi}}=0$ exactly for any case.

Then, recalling that the continuity equation for the Eulerian perturbation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+e^{z/H}\frac{\partial}{\partial z}(e^{-z/H}w)=0, \tag{4.7}$$

we can derive the continuity equation for displacements as

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+e^{z/H}\frac{\partial}{\partial z}(e^{-z/H}\zeta)=0. \tag{4.8}$$

Using this relation, we can rewrite the Stokes velocity vector $\bar{\mathbf{v}}_s$ as

$$\bar{\mathbf{v}}_s=\frac{\partial}{\partial y}(\overline{\eta \mathbf{v}})+e^{z/H}\frac{\partial}{\partial z}(e^{-z/H}\overline{\zeta \mathbf{v}}), \tag{4.9}$$

where $\mathbf{v}$ is (v,w). First, we consider the vertical component $\bar{w}_s$. Since w and ζ are in quadrature in phase as known from (4.6), we have only the first term and it is further rewritten as

$$\bar{w}_s=\frac{\partial}{\partial y}(\overline{\eta w})=-\frac{\partial}{\partial y}(\overline{\zeta v}). \tag{4.10}$$

We now express $\overline{\zeta v}$ in terms of eddy heat flux. Considering the thermodynamic equation for perturbation and rewriting it by using (4.6), we obtain

$$\bar{u}_0\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial z}+\eta\frac{\partial}{\partial y}\left(\frac{\partial \bar{\phi}}{\partial z}\right)+N^2\zeta\right]=0. \tag{4.11}$$

Then, equating the quantity in the brackets to zero and inserting ζ thus obtained into (4.10), we have

$$\bar{w}_s=\frac{1}{N^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial \phi}{\partial z}v}\right). \tag{4.12}$$

This is just the divergence of the eddy heat flux divided by N^2. On the other hand, if we consider a stationary state where no time changes of zonal mean fields occur, we have from (2.5)

$$\bar{w}=-\frac{1}{N^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial \phi}{\partial z}v}\right). \tag{4.13}$$

Thus, in this particular case we see that

$$\bar{w}^{\mathrm{L}}=\bar{w}+\bar{w}_s=0. \tag{4.14}$$

Thus we verify the implication of the CD theorem for the Lagrangian-mean problem, which was mentioned in the Introduction (cf. Uryu, 1974a).

In the present situation, the eddy heat flux is given by (3.4) so that we have

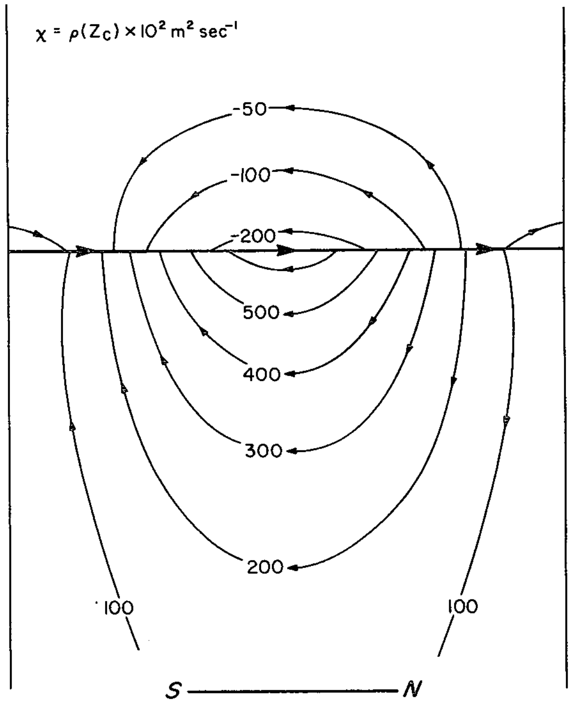

FIG. 5. As in Fig. 3 except for the Lagrangian-mean meridional circulation. A thick solid arrow respresents a concentrated poleward mass flow at the critical level.

$$\bar{w}_s=\frac{\pi Q_0}{fND}e^{\tilde{z}/\tilde{H}}\sin 2\tilde{y}\,\mathrm{H}(-\tilde{z}). \tag{4.15}$$

Comparing this with $\bar{w}$ given by (3.17), we readily obtain $\bar{w}^{\mathrm{L}}$ as

$$\bar{w}^{\mathrm{L}}=-\frac{\pi Q_0}{fND}\sum\frac{1}{\pi\mu_m}\frac{4}{m^2-4}\frac{dG_m}{d\tilde{z}}. \tag{4.16}$$

Recalling (3.16), we notice that $\bar{w}^{\mathrm{L}}$ has a discontinuity at $\tilde{z}=0$, the critical level. This fact suggests that there should be an infinitely strong meridional flow along $\tilde{z}=0$. We shall next consider this meridional flow. Recalling that $iku_0\eta=v$ and using the relationship (4.9), we can write $\bar{v}_s$ as

$$\bar{v}_s=-\frac{1}{N^2}e^{z/H}\frac{\partial}{\partial z}\left(e^{-z/H}\overline{\frac{\partial \phi}{\partial z}v}\right). \tag{4.17}$$

Then using (3.14), we have

$$\bar{v}_s=\frac{\pi Q_0}{f^2D}\sin^2\tilde{y}\,\delta(\tilde{z}). \tag{4.18}$$

Apparently, there is a strong poleward flow concentrated at the critical level. Comparing expression (4.17) with (4.12), we notice that the density-weighted

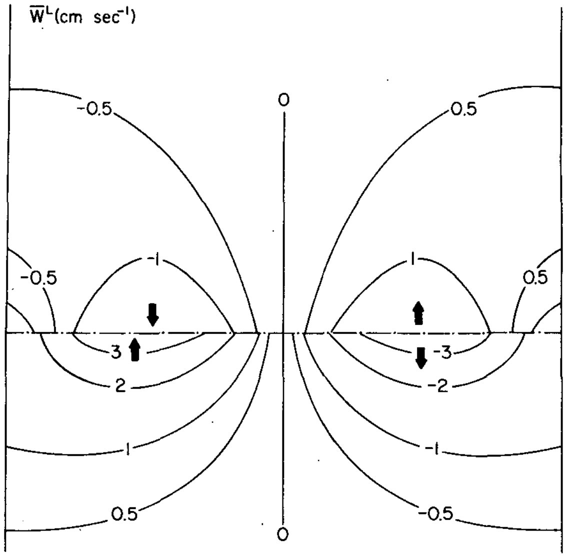

FIG. 6. As in Fig. 4 except for the Lagrangian-mean vertical velocity.

Stokes velocity is solenoidal and hence the Lagrangian-mean mass flux should be nondivergent in the present situation (steady wave). Then the mass streamfunction of the Lagrangian-mean meridional circulation exists and χ_L is obtained from (4.13) as

$$\chi_L = -\bar{\rho}_c \frac{Q_0}{fN} e^{-\tilde{z}/\tilde{H}} \sum \frac{1}{\pi\mu_m} \frac{4}{m(m^2-4)} \frac{dG_m}{d\tilde{z}} \sin m\tilde{y}. \quad (4.19)$$

This function has a jump at $\tilde{z}=0$ consistent with (4.18). The results of numerical calculations of χ_L and $\bar{w}^L$ are shown in Figs. 5 and 6, respectively. In Fig. 5, the discontinuity of the streamfunction is shown by a thick arrow pointing poleward which represents a finite amount of mass flux flowing concentrated in an infinitesimal thin layer. Of course, this is a result of idealization about the structure of the wave at the critical layer. The infinitely strong poleward flow appears to contradict the assumption of small motion. However, if we consider a more realistic situation, the layer may be more spread because of nonstationarity (Dickinson, 1970) and dissipative effects. Then, given a thickness of the layer, the foregoing results may be at least qualitatively correct for waves with an amplitude smaller than some value. In this case, we may consider that a finite but strong poleward movement of individual air particles takes place near the critical level. Further, we may consider that a similar flow occurs near the leading edge of an upward propagating wave packet, if we apply the result to the transient state case (cf. Matsuno, 1971). As seen from Fig. 5, this strong poleward flow bifurcates at the polar boundary to form a downward motion below the critical level and an upward motion above it. At lower latitudes, confluence toward the meridional flow from both above and below occurs. These Lagrangian-mean vertical motions may be regarded as the sole cause of the zonal mean temperature change shown in Fig. 2.

Comparing Figs. 5 and 6 with their Eulerian counterparts (Figs. 3 and 4), we recognize that the two circulations are quite different. Vertical motions below the critical level are in opposite directions. Another interesting difference is that the Lagrangian-mean circulation is localized in the vicinity of the critical level, while the Eulerian-mean circulation pattern is not closed but extends far down to the wave source region. These differences give us different pictures of the mechanism of a sudden warming, as discussed in Section 6.

From (4.12) and (4.17), we understand that the eddy heat flux divided by N^2 plays the role of mass streamfunction of the Stokes velocity. It has been pointed out by Holton (1974) that the same quantity but with a negative sign becomes the streamfunction of the Eulerian-mean circulation in the situation where the CD theorem holds. Then for general states, the Eulerian streamfunction is often decomposed into two—the eddy heat flux and the departure from it and the latter seems to be small (Andrews and McIntyre, 1976; Boyd, 1976). In considering our results, the departure part

$$\chi - \frac{\bar{p}}{N^2}\left(\overline{\frac{\partial \phi}{\partial z} v}\right)$$

is to within our approximations nothing more than the streamfunction of the Lagrangian-mean circulation χ_L. It is small, in fact, in the case of a slowly varying wave packet as shown by Uryu (1974a), for example. But as can be seen from the results of the present study χ_L is not small in the situation of a sudden warming.

5. A direct way to the Lagrangian-mean circulation

In the preceding sections, we first obtained the Eulerian-mean circulation and then computed the Lagrangian-mean by adding the Stokes drift to the former. Recently, a system of equations which directly determine the Lagrangian-mean circulation in the presence of wave disturbances has been derived by Andrews and McIntyre (1978b). Nakamura (1979) also treated mean motions associated with waves by considering energy and momentum balances of individual material parts of a fluid. In what follows, we show that we can get the same results as obtained in the preceding section by applying the equations of Andrews and McIntyre. It should be noted here that, although we follow their treatment, we will make many simplifications by confining ourselves to the particular situation we are now concerned with, i.e., a steady wave (at least up to the second order of wave amplitude).

For simplicity, we consider a Boussinesq fluid, whose mean density is taken as unity and the deviation from it is denoted by ρ. Then, referring to the above authors, we can write the equations of motion and the density conservation equation for means over a material tube as follows:

$$\tilde{\rho}\left[\left(\frac{D}{Dt}\right)_L \bar{\mathbf{v}}^L + f\mathbf{k}\times\bar{\mathbf{v}}^L + \mathbf{k}\times\beta\overline{\eta\mathbf{v}}\right] = -\nabla\bar{p}^L + \mathrm{div}\mathbf{R} - \bar{\rho}^L g\mathbf{k}, \quad (5.1)$$

$$\left(\frac{D}{Dt}\right)_L \bar{\rho}^L = 0, \quad (5.2)$$

where we write

$$\left(\frac{D}{Dt}\right)_L \equiv \frac{\partial}{\partial t} + \bar{\mathbf{v}}^L\cdot\nabla \quad (5.3)$$

and $\mathbf{k}$ is the vertical unit vector. Further, $\tilde{\rho}$ is the "mean flow density" whose significance will be described later. $\mathbf{R}$ is the radiation stress (Bretherton, 1971; Andrews and McIntyre, 1978b) and is written correct to the second order of wave amplitude as

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2, \quad (5.4)$$

where $\mathbf{R}_1$ is

$$\mathbf{R}_1 = \begin{bmatrix} \overline{p\frac{\partial\xi}{\partial x}} & \overline{p\frac{\partial\eta}{\partial x}} & \overline{p\frac{\partial\zeta}{\partial x}} \\ \overline{p\frac{\partial\xi}{\partial y}} & \overline{p\frac{\partial\eta}{\partial y}} & \overline{p\frac{\partial\zeta}{\partial y}} \\ \overline{p\frac{\partial\xi}{\partial z}} & \overline{p\frac{\partial\eta}{\partial z}} & \overline{p\frac{\partial\zeta}{\partial z}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_x^T \\ \mathbf{R}_y^T \\ \mathbf{R}_z^T \end{bmatrix} \quad (5.5)$$

and $\mathbf{R}_x^T$, $\mathbf{R}_y^T$ and $\mathbf{R}_z^T$ are row vectors consisting of the (first part of) the radiation stress tensor. In later discussion, we denote their transposes (therefore column vectors) by omitting T. $\mathbf{R}_2$ is that part of the radiation stress which arises from the second-order terms of Taylor expansions of the basic state pressure gradient, such as $\frac{1}{2}[\overline{\xi\xi}]:[\nabla\nabla]\partial p_0/\partial y$ which are not important in our problem, as will be shown later, so that their explicit forms need not be written here.

By treating Lagrangian-mean quantities, we do not have "eddy transport" effects, because we are considering a definite material tube of fluid, into which no mass, hence no momentum or density (potential temperature) is carried in or taken out by fluid motion. Instead, we have "radiation stress," which is essentially the pressure drag exerted across an undulating material surface which encloses the tube (see Bretherton, 1971; Andrews and McIntyre, 1978b). It arises from correlations between surface slope and disturbance pressure.

The continuity equation for the Lagrangian-mean motion has a somewhat unfamiliar character. Namely, even though the fluid is incompressible, the motion of centers of mass of material tubes projected onto the meridional plane becomes divergent, in general (McIntyre, 1973; Andrews and McIntyre, 1978b; Nakamura, 1979). For example, when a wave is growing, material tubes make larger and larger undulations and overlapping of neighboring material tubes occurs as seen on the meridional plane. Hence, their centers of mass tend to converge toward the common center of mass (Nakamura, 1979).

Thus, following Andrews and McIntyre (1978b), we introduce a dimensionless mean flow density $\tilde{\rho}$ which expresses the degree of overlapping of material tubes, in order to ensure mass conservation for the Lagrangian-mean flow. They have shown that $\tilde{\rho}$ is given (correct to the second order of wave amplitude) as

$$\tilde{\rho} = 1 - \tfrac{1}{2}[\nabla\nabla]:[\overline{\xi\xi}] \quad (5.6)$$

and also that the relation

$$\nabla\cdot\bar{\mathbf{v}}^L = \frac{1}{2}\left(\frac{D}{Dt}\right)_L [\nabla\nabla]:[\overline{\xi\xi}] \quad (5.7)$$

holds. Then they were able to write the mass conservation equation as

$$\left(\frac{D}{Dt}\right)_L \tilde{\rho} + \tilde{\rho}\nabla\cdot\bar{\mathbf{v}}^L = 0. \quad (5.8)$$

We now apply these equations to obtaining $\bar{v}^L$ and $\bar{w}^L$ correct to second order in the situation under consideration. As the basic state, we have considered a stationary zonal wind field u_0, which is a function of z, and the shear is in thermal wind balance with the basic-state density gradient $\partial\rho_0/\partial y$. Further, we have assumed that a stationary planetary wave is set up in the basic zonal wind field. Then the wave causes changes in the zonal mean fields as well as the second-order flow by its non-linear effects. What we are now concerned with are these second-order flows averaged in the Lagrangian sense. The equations for these quantities are written as

$$\frac{\partial\bar{u}^L}{\partial t} + \bar{v}^L\frac{\partial u_0}{\partial y} + \bar{w}^L\frac{\partial u_0}{\partial z} - f\bar{v}^L - \beta\overline{\eta v} = \nabla\cdot\mathbf{R}_x, \quad (5.9)$$

$$\frac{\partial\bar{v}^L}{\partial t} + f\bar{u}^L + \beta\overline{\eta u} = -\frac{\partial\bar{p}^L}{\partial y} + \nabla\cdot\mathbf{R}_y, \quad (5.10)$$

$$\frac{\partial\bar{w}^L}{\partial t} = -\frac{\partial\bar{p}^L}{\partial z} + \nabla\cdot\mathbf{R}_z - \bar{\rho}^L g, \quad (5.11)$$

$$\frac{\partial\bar{\rho}^L}{\partial t} + \bar{v}^L\frac{\partial\rho_0}{\partial y} + \bar{w}^L\frac{\partial\rho_0}{\partial z} = 0, \quad (5.12)$$

$$\frac{\partial\bar{v}^L}{\partial y} + \frac{\partial\bar{w}^L}{\partial z} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}[\nabla\nabla]:[\overline{\xi\xi}]\right). \quad (5.13a)$$

Here we have made approximations such as

$$\frac{\partial}{\partial t}+\bar{v}^{\mathrm{L}}\frac{\partial}{\partial y}+\bar{w}^{\mathrm{L}}\frac{\partial}{\partial z}=\frac{\partial}{\partial t}+\mathrm{O}(a^2),$$

where $\mathrm{O}(a^n)$ means a quantity of nth order of the wave amplitude a. We have already made approximations which are correct up to second order of wave amplitude in the definition of the Lagrangian-mean and in deriving (5.1)–(5.8). We have also dropped the terms corresponding to $\mathrm{div}\mathbf{R}_2$ for the following reason. In the basic state, both the hydrostatic and geostrophic balances hold at any place. Then, by Taylor-expanding the both sides of the two equations, we have

$$[\overline{\xi\xi}]:[\nabla\nabla]fu_0=-[\overline{\xi\xi}]:[\nabla\nabla]\frac{\partial p_0}{\partial y}, \quad (5.14a)$$

$$-g[\overline{\xi\xi}]:[\nabla\nabla]\rho_0=[\overline{\xi\xi}]:[\nabla\nabla]\frac{\partial p_0}{\partial z}. \quad (5.14b)$$

Then we may consistently omit those terms which came from the second order of Taylor expansions of the basic field variables. In effect, this is equivalent to modifying the definition of the Lagrangian-means as

$$\bar{B}^{\mathrm{L}}=\bar{B}+\overline{\xi\cdot\nabla b}, \quad (5.15)$$

where B is u, v, w, p or ρ. Further, by recalling that the wave is steady up to $\mathrm{O}(a^2)$, so its change is at most $\mathrm{O}(a^3)$, we see that the continuity equation (5.13a) becomes

$$\frac{\partial\bar{v}^{\mathrm{L}}}{\partial y}+\frac{\partial\bar{w}^{\mathrm{L}}}{\partial z}=\mathrm{O}(a^4)\approx 0. \quad (5.13b)$$

Because of stationarity of the wave, η is 90° out of phase with v, so that the $\beta\overline{\eta v}$ term in (5.9) can be neglected, too.

In the present problem, the hydrostatic and the quasi-geostrophic approximations are permissible. Thus, making these approximations, we have

$$\frac{\partial}{\partial t}\bar{u}^{\mathrm{L}}-f\bar{v}^{\mathrm{L}}=\nabla\cdot\mathbf{R}_x, \quad (5.16)$$

$$f\bar{u}+\beta\overline{\eta u}=-\frac{\partial\bar{p}^{\mathrm{L}}}{\partial y}+\nabla\cdot\mathbf{R}_y, \quad (5.17)$$

$$0=-\frac{\partial\bar{p}^{\mathrm{L}}}{\partial z}+\nabla\cdot\mathbf{R}_z-\bar{\rho}^{\mathrm{L}}g, \quad (5.18)$$

$$\frac{\partial}{\partial t}\bar{\rho}^{\mathrm{L}}+\bar{w}^{\mathrm{L}}\frac{\partial\rho_0}{\partial z}=0, \quad (5.19)$$

$$\frac{\partial\bar{v}^{\mathrm{L}}}{\partial y}+\frac{\partial\bar{w}^{\mathrm{L}}}{\partial z}=0. \quad (5.20)$$

In accord with the geostrophic approximations and the assumption of a midlatitude β plane, we shall regard f as a constant hereafter. This set of equations can be regarded as the Lagrangian version of (2.2)–(2.5). Now from the last condition there exists a streamfunction χ_{L} of the Lagrangian-mean meridional circulation. Then, expressing $\bar{v}^{\mathrm{L}}$ and $\bar{w}^{\mathrm{L}}$ in terms of χ_{L} and by eliminating $\bar{\rho}^{\mathrm{L}}$, $\bar{p}^{\mathrm{L}}$ and $\bar{u}^{\mathrm{L}}$ by use of the thermal wind relationship, we can get a diagnostic equation to determine χ_{L}, in almost the same way as we do in the usual Eulerian-mean circulation problem. But some particular care is necessary in the course of manipulation because differentiation and the Lagrangian-mean operation do not comute, e.g.,

$$\frac{\partial}{\partial z}\left(\frac{\partial\bar{p}^{\mathrm{L}}}{\partial y}\right)\neq-\frac{\partial\bar{\rho}^{\mathrm{L}}}{\partial y}g,$$

for example. Fortunately, in our problem this difficulty is avoided by virtue of the fact that time change of the wave structure is $\mathrm{O}(a^3)$ as mentioned previously. Namely, in operating $\partial/\partial t$ on both sides of (5.17), we have

$$\frac{\partial\bar{u}^{\mathrm{L}}}{\partial t}=\frac{\partial\bar{u}}{\partial t}+\overline{\frac{\partial\xi}{\partial t}\cdot\nabla u}+\overline{\xi\cdot\nabla\frac{\partial u}{\partial t}}$$

$$=\frac{\partial\bar{u}}{\partial t}+\mathrm{O}(a^4),$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial\bar{p}^{\mathrm{L}}}{\partial y}+\nabla\cdot\mathbf{R}_y\right)=\frac{\partial}{\partial t}\left[-\frac{\partial\bar{p}}{\partial y}-\frac{\partial}{\partial y}(\overline{\xi\cdot\nabla p})+\nabla\cdot\overline{\left(p\frac{\partial\xi}{\partial y}\right)}\right]$$

$$=-\frac{\partial^2\bar{p}}{\partial t\partial y}+\mathrm{O}(a^4).$$

Since we expect that the leading order of $\partial\bar{u}^{\mathrm{L}}/\partial t$ is a^2, in accordance with $\bar{v}^{\mathrm{L}}$, $\mathbf{R}_x\sim\mathrm{O}(a^2)$ in (5.9), we neglect $\mathrm{O}(a^4)$ terms. Thus, we finally obtain the χ_{L} equation in the form

$$N^2\frac{\partial^2\chi_{\mathrm{L}}}{\partial y^2}+f^2\frac{\partial^2\chi_{\mathrm{L}}}{\partial z^2}=f\frac{\partial}{\partial z}(\nabla\cdot\mathbf{R}_x). \quad (5.21)$$

Now for the sake of comparison with the Eulerian streamfunction equation, we rewrite the right-hand side as

$$\mathbf{R}_x=\frac{1}{u_0}\overline{pu_0\frac{\partial\xi}{\partial x}}=\frac{1}{u_0}\overline{p\mathbf{v}}. \quad (5.22)$$

Further, in the case of quasi-geostrophic motion, we have the relations derived by Eliassen and Palm (1961):

$$\frac{\overline{pv}}{u_0}=-\overline{uv},$$

$$\frac{\overline{pw}}{u_0}=\frac{f}{N^2}\overline{\left(\frac{\partial p}{\partial z}v\right)}.$$

Then (5.21) is rewritten as

$$\mathcal{L}(\chi_L)=\frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\overline{\left(\frac{\partial p}{\partial z}v\right)}-f\frac{\partial^2}{\partial z\partial y}(\overline{uv}), \qquad (5.23)$$

where $\mathcal{L}$ stands for the Laplacian-type operator appearing in (5.21). On the other hand, the equation determining the stream function χ of the Eulerian-mean meridional circulation is derived from (2.2)–(2.5), after similar manipulation, as

$$\mathcal{L}(\chi)=-\frac{\partial^2}{\partial y^2}\overline{\left(\frac{\partial p}{\partial z}v\right)}-f\frac{\partial^2}{\partial z\partial y}(\overline{uv}). \qquad (5.24)$$

Comparing the two equations, we immediately notice that the eddy heat flux enters in the right-hand side in different forms, thus causing different circulations. The momentum flux appears in exactly the same way, although it is zero in the particular situation we are now considering. Recalling the functional form of the eddy heat flux (3.4) (precisely speaking, its Boussinesq version), we have explicit forms of the two equations in the present situation:

Lagrangian χ_L equation

$$\mathcal{L}(\chi_L)=C_1\sin^2\frac{\pi}{D}y\delta'(z_c-z).$$

Eulerian χ equation

$$\mathcal{L}(\chi)=C_2\cos\frac{2\pi}{D}y\mathrm{H}(z_c-z).$$

In the above, the C's are constants and $\delta'(z)=(d/dz)\delta(z)$. Solving the two equations under suitable boundary conditions, we can recover χ and χ_L obtained previously, except that we must set $H=\infty$ in those solutions.

6. Lagrangian view on the sudden warming mechanism

In the preceding section, we discussed the Lagrangian-mean meridional circulation directly based on the system of equations governing the Lagrangian-mean fields, assuming steady small-amplitude waves. By considering physical meanings of the equations, we are led to an alternative view on the mechanism of a sudden warming different from the one proposed by Matsuno (1971) and based on the Eulerian-mean equations.

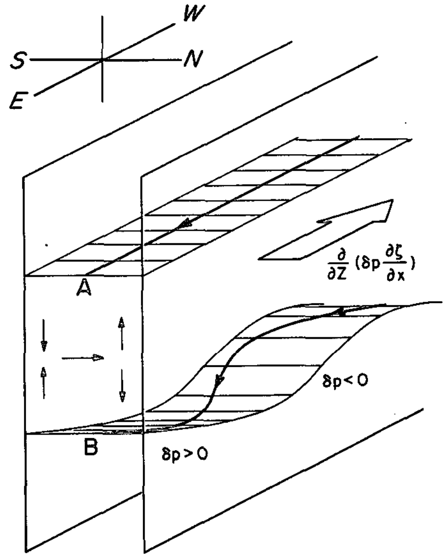

FIG. 7. Schematic illustration of the mechanism of a sudden warming from the Lagrangian viewpoint. See text for futher explanation.

First, we consider the Lagrangian χ_L equation in its original form (5.21), comparing it with the Eulerian counterpart (5.24). In the χ equation, we have divergences of both eddy heat flux and eddy momentum flux as forcing terms, and we interpret that the circulation is caused by response to thermal as well as mechanical forcings. On the contrary, in the χ_L equation we have only divergence of radiation stress on its right-hand side so that χ_L is considered to be caused by the wave only through its mechanical effects. The contrast is most clear in the situation treated in this study, because the Eulerian circulation is entirely due to eddy heat transport. Matsuno (1971) discussed how this heat transport causes the meridional circulation and then how the easterly acceleration of the zonal flow and corresponding temperature changes occur, when a planetary wave is incident on a critical level or when the wave is in a transient state of upward propagation. We now consider the same problem from the viewpoint of the Lagrangian-mean motion. In Fig. 7, the configuration of the wave and the zonal flow is shown schematically. For qualitative discussion, it does not matter whether we consider a critical level situation or wave transience and therefore the figure is drawn assuming the latter situation.[5] We assume that the planetary

[5] The central point of the subsequent arguments is the Lagrangian-mean circulation in the meridional plane caused by the gradient of the zonal component of radiation stress. Though wave amplification may cause a convergent meridional flow in the Lagrangian-mean sense at the leading edge of the wave packet (cf. Nakamura, 1979), we are not now concerned with this part of motion.

wave which has propagated from below has reached a lower surface designated as B, but that does not affect an upper surface A, both of which are supposed to be material surfaces. The surface B, which was originally a flat surface is now undulating under the presence of the wave. The wave structure, i.e., the phase differences among the vertical displacement of the surface, the pressure perturbation δp (which is the deviation from the undisturbed value at the current height) and the perturbation meridional velocity are determined from the first-order perturbation equation and shown schematically. A typical streamline is also drawn. As indicated on the figure, the pressure perturbation is positive at the eastward up-slope and negative at eastward downslope so that we see that

$$(\mathbf{R}_x)_z=\overline{p\frac{\partial\zeta}{\partial x}}>0.$$

Physically, it means that the air existing below the surface B exerts westward force on the air above B. The same force is acting through any surfaces in the fluid below B because we presume that the wave has settled down to a steady state there and in this case

$$\frac{\partial}{\partial z}\left(\overline{p\frac{\partial\zeta}{\partial x}}\right)=\frac{\partial}{\partial z}\left(\frac{\overline{pw}}{u_0}\right)=0$$

(Eliassen and Palm, 1961). But in the fluid layer between the surfaces A and B, where we have the leading edge of the wave, the component of the radiation stress decreases with height in magnitude. Then we understand a net westward (easterly accelerating) force appears in this part. The force is essentially the same as the pressure drag exerted by the solid body when we solidify the lower portion of the fluid below the surface B, and therefore it is the same as the force given to the atmosphere from an undulating ground surface located far below, if the wave is generated by orography (cf. Uryu, 1974b; McIntyre, 1977). The response of the atmosphere to this westward force is determined by the χ_L Eq. (5.24), the solution of which has been obtained and shown in Fig. 5. Let us interpret the result physically. Originally, westerly winds are prevailing in this part of the atmosphere and therefore the zonal mean pressure is lower on the poleward side. When the part undergoes westward acceleration, the balance between the pressure gradient and the Coriolis force no longer holds (the latter becomes weaker) so that a poleward flow should appear. In short, a cross-isobaric flow (the Ekman transport) appears in response to the resisting force to the zonal flow. This is the strong poleward flow that appeared in Fig. 5. As a result of this poleward-converging flow, there should occur vertical motions, as shown in Fig. 5 and also shown schematically in the meridional section in Fig. 7. Unlike the frictional convergence in the near-ground layer, vertical motion in this uplifted friction layer, so to speak, can occur in either direction. Corresponding to the four branches of the Lagrangian-mean vertical motions, warmings and coolings take place in four sectors of the meridional section, as seen in Fig. 2. In the real atmospheric situation, the warming on the poleward side is most pronounced because of the earth's sphericity and because of the compressibility of the air.

Recently, Holton and Dunkerton (1978) raised a question whether a *steady-state* critical level could be a model of the sudden warming situation. By analyzing the results of the numerical experiments by Holton and Mass (1976), they have shown that the wave transience effect is dominating in potential vorticity balance near the critical level, when a warming is taking place. Though we did not treat potential vorticity balance explicitly in the present study, the delta-function-shaped forcing in the zonal mean potential vorticity equation represents the southward transport of potential vorticity at the critical level, which arises from large meridional dispersion of potential vorticity at the level, as discussed by Bretherton (1966) and Dickinson (1969). Therefore, in this sense the effect of wave transience is included in the present study. Corresponding to the dispersion of air particles there must be a converging meridional flow in the Lagrangian-mean motion at the critical level, but this feature escaped from our analysis because of the steady-state assumption which fails to determine displacements at the level.

7. Summary and remarks

In the present study, we have discussed two types of mean meridional circulations. One is the usual zonal mean along a constant latitude-height line (the Eulerian mean); the other is the mean along a wavy material line consisting of a definite member of individual air particles, which would coincide with a latitude circle, were the wave not present (the Lagrangian mean). The latter was obtained by adding the Stokes drift to the former.

We treated circulations which arise when an upward-propagating planetary wave is incident on a critical level, a model situation of the sudden warming. The Eulerian-mean circulation calculated for this situation is essentially the same as that described by Matsuno (1971) based on qualitative discussion. Namely, an upward motion appears at higher latitudes, a downward motion at lower latitudes, and an equatorward meridional flow in the vicinity of the critical level. The Lagrangian-mean circulation is quite different from this Eulerian circulation, implying that the contribution from the Stokes drift is quite large, just as found by Uryu (1974a) and Andrews and McIntyre (1976) in related problems. A strong poleward meridional flow occurs concentrated at the critical level. This meridional flow is bifurcated at the boundary to form on the poleward side a downward motion below the critical level and an upward motion above the level. On the lower latitude side, confluent vertical motions toward the meridional flow axis occur in an opposite way to those on the higher latitude side. Thus, the Lagrangian-mean vertical motion field has a four-sector structure which is reflected in the zonal mean temperature changes.

The origin of the strong poleward flow at the critical level can be understood by considering the dynamics of material parts of fluid, based on the equations governing their motions (Andrews and McIntyre, 1978b). Applying these equations to the present problem, we obtained an equation to determine the Lagrangian-mean meridional circulation, in which the forcing term to cause the circulation is a westward force at the critical level. The force comes from a sharp cutoff of the zonal component of the radiation stress there. Thus, a strong poleward flow results in the same way as the Ekman transport in ocean currents or a cross-isobaric flow in the frictional layer. Thus, we can get an alternative view on the mechanism of the sudden warming, which is complementary to the explanation given by Matsuno (1971) from the Eulerian viewpoint.

The Lagrangian-mean motion is suited for the understanding of movement of trace substances. The results of the present study indicate that pronounced poleward movement of trace substances would take place near the critical level, on the occasion of a sudden warming or in other critical layer situations. Since counter flows appear at other levels, the vertically integrated transport depends on the distribution of the substance. At the time of a spring reversal of the stratospheric zonal flow, a critical layer exists in the middle stratosphere where ozone concentration is maximum. Then we may consider that considerable amounts of ozone is transported poleward. By analyzing the heat balance of a general circulation model, Mahlman and Moxim (1978) found that a considerable amount of net heating due to dynamical effects exists at the 65 mb level in July, and they conjectured that it may be a critical level effect. This result is consistent with ours.

The present study deals with the Lagrangian *mean* motion, namely, motion of the center of mass of many individual air particles. Dispersion of the particles around their center of mass is another important problem concerning the transport of trace substances (cf. Kao *et al.*, 1976; Kida, 1977). Since meridional diffusion of air particles is supposed to become very large at a critical level (Bretherton, 1966; Kida, 1977), we must keep this in mind in applying the present results to interpretation of motion of trace substances at the time of a sudden warming.

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