

Generalized Eliassen-Palm and Charney-Drazin Theorems for Waves on Axisymmetric Mean Flows in Compressible Atmospheres

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ABSTRACT

The theorems, which exhibit the role of wave dissipation, excitation and transience in the forcing of mean flow changes of second order in wave amplitude by arbitrary, small-amplitude disturbances, are obtained 1) for the primitive equations in pressure coordinates on a sphere, and 2) in a more general form (applicable for instance to nonhydrostatic disturbances in tornadoes or hurricanes) establishing that no approximations beyond axisymmetry of the mean flow are necessary. It is shown how the results reduce to those found by Boyd (1976) for the case of sinusoidal, hydrostatic waves with exponentially growing or decaying amplitude, and it is explained why the approximation used by Boyd in the thermodynamic equation is not needed. The reduction to Boyd's results entails the use of a virial theorem. This theorem amounts to a generalization of the "equipartition" law derived in an earlier paper (Andrews and McIntyre, 1976). That derivation appeared to rely on an assumption about relative phases of disturbance Fourier components; the present derivation shows that no such assumption is in fact necessary.

1. Introduction

In a recent paper (Andrews and McIntyre, 1976; hereafter referred to as I) we derived some new results on wave, mean-flow interaction (the generalized Eliassen-Palm and Charney-Drazin theorems), and used them to discuss the effect of wave dissipation mechanism upon the profile of the mean zonal acceleration forced by equatorial planetary waves.

The theorems are of far wider applicability, however, and it is of interest to know to what extent they depend on various approximations, and exactly how they reduce to the results of Boyd (1976); the purpose of this paper is to answer both these questions, so far as seems practicable within the conventional analytical description of wave, mean-flow interaction. Thus Section 6 derives the theorems under no restrictions other than that the mean flow be axisymmetric and the disturbance amplitude small; in particular, the hydrostatic and "traditional" approximations are not required. The theorems are obtained in a vectorial form which is equally easy to realize in cylindrical or spherical coordinates. As well as being of fundamental interest, the results of Section 6 are directly applicable, for example, to the disturbances to tornadoes discussed by Stern (1971) and to wave models of hurricane rainbands (Willoughby, 1977, and references), both of which involve nonhydrostatic inertio-gravity waves. They are also applicable to acoustic-gravity waves, tides, and to wave motions in stellar interiors. The re-

striction to small disturbance amplitude can be lifted as well (Andrews and McIntyre, 1977), but this will not be pursued here, since it requires use of a novel "generalized Lagrangian-mean" description of the wave, mean-flow interaction problem, whose practical limitations, especially near critical lines, are not well understood as yet. Section 6 can be read independently of Sections 3-5. At the end of Section 6 we refer to connections, recently brought to light by the independent analysis of Bretherton (1977), between our results and the notions of "pseudomomentum" and "wave action." But we emphasize that the derivations in I and in the present paper are elementary, and do not presume any familiarity with wave-action concepts and related ideas.

For use with observational and theoretical studies of the general circulation, hydrostatic versions of the theorems based on the primitive equations in pressure coordinates on the sphere are relevant, and these are presented in Section 3. Such versions are in principle contained in the more general results of Section 6, but the relationship is not straightforward because a zonal average at constant pressure differs from one at constant height. It is simpler to rework the derivation for the new set of equations. Since the analysis for pressure coordinates is formally almost identical to that for the Boussinesq equations given in I, only a few additional details are required here. [A fuller exposition and motivation of the basic analysis, together with a careful discussion of various matters of physical interpretation, is given in the recent review article by McIntyre (1977).]

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In Sections 4 and 5 we indicate how these pressure-coordinate results reduce to those of Boyd (1976) when the disturbance fields are taken to have the special form assumed by him, namely

$$\text{Re}\{\text{func}(\varphi, p) \exp(bt) \exp(im\lambda - i\omega t)\}, \quad (1.1)$$

where (λ, φ, p) are longitude, latitude and pressure, and $b, m,$ and ω are real constants. The reduction is not trivial, and involves the use of a "virial theorem" obtained by taking the scalar product of the disturbance particle displacements with the disturbance momentum equation (Section 4). The virial theorem also leads to a strengthened statement of the subsidiary results (7.4) and (7.5) of I. It shows that they do not, contrary to what we thought earlier, depend on any assumption about the relative phases of disturbance Fourier components. Virial theorems of this sort are generalizations of the classical "equipartition of energy" theorems which hold for some very simple types of wave. As mentioned in I, such theorems can furnish a very useful check on numerical calculations of complicated wave structures.

Our alternative derivation of Boyd's results reveals a further point of some interest; namely, that they hold, like all the results of the present paper, without the additional approximation made by Boyd in the thermodynamic equation (Holton 1975, p. 32). In Section 5 we indicate briefly how Boyd's analysis can itself be modified so as not to rely on that approximation. Of course the fact that the theorems also hold under the additional approximation (we have checked that this is true of our results as well as Boyd's) is a significant statement about the approximate equations themselves. Presumably we should add consistency with the generalized Eliassen-Palm relations to classical criteria such as "energetic consistency" (e.g. Lorenz, 1967, 15-23), when assessing the status of any set of approximated equations as a model of physical reality. In this connection we mention that the sigma-coordinate equations often used in numerical modeling (e.g. Phillips 1973, p. 24) do in fact satisfy this criterion; they also possess the generalized Eliassen-Palm relations. (We omit the details, but would be willing to supply them personally to anyone interested.) It appears from the work of Bretherton (1977) that a given set of equations will possess generalized Eliassen-Palm relations whenever the corresponding conservative equations possess a Kelvin circulation theorem, for any material circuit which would take the form of a zonal ring in the absence of disturbances.

2. Equation of state

Throughout this paper we deal with an arbitrary compressible fluid. Let ρ be density and p pressure. The equation of state will be written in the general form

$$\rho = F(\theta, p), \quad (2.1)$$

where θ is potential temperature (or entropy, with suitable redefinition of the function F). For the case of a perfect gas with specific heat ratio γ , we have

$$F(\theta, p) = \rho_\infty (T_\infty/\theta) (p/p_\infty)^{1/\gamma}, \quad (2.2)$$

if θ is taken as potential temperature relative to some reference state with temperature T_∞ and with $\rho = \rho_\infty, p = p_\infty$.

Let $(\bar{\quad})$ denote an averaging operator and $(\quad)'$ the associated fluctuation, defined as $(\quad) - (\bar{\quad})$, which will be considered throughout to be $O(a)$ for some small amplitude parameter a . By definition, $(\bar{\quad})' = 0$. Let

$$\mathcal{F} \equiv F(\bar{\theta}, \bar{p}), \quad (2.3)$$

$$\mathcal{F}_{\bar{\theta}} \equiv \partial F(\theta, p) / \partial \theta |_{\theta=\bar{\theta}, p=\bar{p}}, \quad (2.4)$$

and similarly $\mathcal{F}_{\bar{\theta}\bar{\theta}}, \mathcal{F}_{\bar{\theta}\bar{p}},$ etc. Then, expanding (2.1) about the mean values, we have

$$\rho' = \mathcal{F}_{\bar{\theta}} \theta' + \mathcal{F}_{\bar{p}} p' + O(a^2), \quad (2.5)$$

$$\bar{p} - \mathcal{F} = \frac{1}{2} (\mathcal{F}_{\bar{\theta}\bar{\theta}} \overline{\theta'^2} + 2\mathcal{F}_{\bar{\theta}\bar{p}} \overline{\theta' p'} + \mathcal{F}_{\bar{p}\bar{p}} \overline{p'^2}) + O(a^3). \quad (2.6)$$

For later use we define

$$\mathcal{G} \equiv - \frac{\partial}{\partial \theta} \frac{1}{F(\theta, p)} \Big|_{\theta=\bar{\theta}, p=\bar{p}} = \frac{\mathcal{F}_{\bar{\theta}}}{\mathcal{F}^2} \quad (2.7)$$

All the above quantities remain finite in the incompressible limit $F(\theta, p) = F(\theta)$.

3. Extension of the results of I to pressure coordinates

In spherical geometry, the primitive equations in pressure coordinates under the conventional hydrostatic and "traditional" approximations are (e.g. Lorenz 1967; Phillips 1973):

$$u_t + R^{-1}(u^2)_\lambda + \frac{1}{r_0 \cos^2 \varphi} (vu \cos^2 \varphi)_\varphi + (\omega u)_p - fv + R^{-1}\Phi_\lambda = -X, \quad (3.1a)$$

$$v_t + R^{-1}(uv)_\lambda + R^{-1}(v^2 \cos \varphi)_\varphi + (\omega v)_p + fu + \frac{u^2 \tan \varphi}{r_0} + \frac{1}{r_0} \Phi_\varphi = -Y, \quad (3.1b)$$

$$1/F(\theta, p) + \Phi_p = 0, \quad (3.1c)$$

$$\theta_t + R^{-1}(u\theta)_\lambda + R^{-1}(v\theta \cos \varphi)_\varphi + (\omega \theta)_p = -Q, \quad (3.1d)$$

$$R^{-1}u_\lambda + R^{-1}(v \cos \varphi)_\varphi + \omega_p = 0, \quad (3.1e)$$

where

$$R = r_0 \cos \varphi,$$

r_0 being the radius of the earth. The independent variables (λ, φ, p) denote longitude, latitude and pressure, and $(u, v, \omega) \equiv (RD\lambda/Dt, r_0 D\varphi/Dt, Dp/Dt)$ the corresponding velocity components. Φ is the geopotential, equal to height times a constant gravitational acceleration, and X, Y, Q represent dissipation or forcing terms, whose form is left unspecified. F is the function of state appearing in (2.1), and f the Coriolis parameter.

With the exception of (3.1c), this set of equations is formally identical to the Boussinesq, hydrostatic set used in I, with ω, p and Φ in place of w, z or r , and p . In the derivation of pressure-coordinate results analogous to I(5.5) or I(5.9), the only point needing extra attention is the fact that the first term in (3.1c) depends on p as well as on θ ; in I the corresponding term is just $-\theta$ (i.e., $F \equiv -\theta^{-1}, \mathcal{G} \equiv 1$). The dependence on p gives rise to an extra term involving the Lagrangian disturbance pressure p^l , defined in (3.10a) below (in pressure coordinates, p^l replaces the vertical displacement ζ' appearing in I). But that extra term in p^l disappears from the final result, because in the course of the analysis it is zonally averaged after multiplication by its own zonal derivative [cf. (A13)]. Since the contribution from the first term in the Boussinesq analogue of (3.1c) gives rise to just one term in I(5.9a), namely $R^{-1}\xi_\lambda \bar{q}'$, that term is here replaced by $R^{-1}\mathcal{G} p^l \bar{q}'$ where \mathcal{G} , defined in (2.7), involves the partial derivative of F with respect to θ only. The same, of course, holds for Cartesian pressure coordinates (x, y, p) , with $R^{-1}\partial/\partial\lambda$ replaced by $\partial/\partial x$.

For completeness, we state the full set of results, starting with the transformed mean-flow problem analogous to that given in I. Let $(\bar{\quad})$ now denote the zonal average at constant p . If we define \bar{v}^* and $\bar{\omega}^*$ by

$$\bar{\omega} = \bar{\omega}^* - R^{-1} \left(\frac{v'\theta'}{\bar{\theta}_p} \cos \varphi \right)_\varphi, \quad \bar{v} = \bar{v}^* + \left(\frac{v'\theta'}{\bar{\theta}_p} \right)_p, \quad (3.2)$$

and

$$S_{(\lambda\varphi)} = \bar{v}'u' - \mathcal{B}v'\theta'/\bar{\theta}_p, \quad (3.3a)$$

$$S_{(\lambda p)} = \bar{\omega}'u' + \mathcal{A}v'\theta'/\bar{\theta}_p, \quad (3.3b)$$

where

$$\mathcal{A} \equiv R^{-1}(\bar{u} \cos \varphi)_\varphi - f = R^{-1}r_0^{-1}\mathfrak{N}_\varphi, \quad (3.4)$$

$$\mathcal{B} \equiv \bar{u}_p = R^{-1}\mathfrak{N}_p, \quad (3.5)$$

$$\mathfrak{N} \equiv r_0 \bar{u} \cos \varphi + r_0^2 \Omega \cos^2 \varphi, \quad (3.6)$$

then the resulting transformed mean-flow problem may be written

$$\begin{aligned} & \bar{u}_t + \mathcal{A}\bar{v}^* + \mathcal{B}\bar{\omega}^* \\ &= -\frac{1}{r_0 \cos^2 \varphi} (S_{(\lambda\varphi)} \cos^2 \varphi)_\varphi - (S_{(\lambda p)})_p - \bar{X}, \end{aligned} \quad (3.7a)$$

$$\begin{aligned} & \bar{v}_t^* + \left(f + \frac{2\bar{u} \tan \varphi}{r_0} \right) \bar{u}_t \\ & + \frac{1}{r_0} \bar{\Phi}_{\varphi t} = -\frac{\partial}{\partial t} \left[-R^{-1}(\bar{v}'^2 \cos \varphi)_\varphi - (\bar{\omega}'v')_p \right. \\ & \quad \left. - \left(\frac{v'\theta'}{\bar{\theta}_p} \right)_{pt} - \bar{Y} + O(a^4) \right], \end{aligned} \quad (3.7b)$$

$$-\mathcal{G}\bar{\theta}_t + \bar{\Phi}_{pt} = -\frac{\partial}{\partial t} [\mathfrak{F}^{-3} (\frac{1}{2} \mathfrak{F} \mathfrak{F}_{\theta\theta} - \mathfrak{F}_{\theta}^2) \bar{\theta}'^2 + O(a^3)] \quad (3.7c)$$

$$\bar{\theta}_t + r_0^{-1} \bar{\theta}_\varphi \bar{v}^* + \bar{\theta}_p \bar{\omega}^* = -\frac{\partial}{\partial p} \left(\bar{\omega}'\theta' + \frac{v'\theta'\bar{\theta}_\varphi}{r_0 \bar{\theta}_p} \right) - \bar{Q}, \quad (3.7d)$$

$$R^{-1}(\bar{v}^* \cos \varphi)_\varphi + \bar{\omega}_p^* = 0. \quad (3.7e)$$

[Note that the right-hand side of (3.7c) would be negligible in most conceivable applications, and that in any case the $O(a^2)$ contribution vanishes for a perfect gas since by (2.2) and (2.3) \mathfrak{F} is then proportional to $\bar{\theta}^{-1}$.]

Eqs. (3.7) comprise a complete set of equations for the $O(a^2)$ quantities

$$\{\bar{u}_t, \bar{\theta}_t, \bar{\Phi}_t, \bar{v}^*, \bar{\omega}^*\}. \quad (3.8)$$

The "coefficients" $\mathcal{A}, \mathcal{B}, (f + 2r_0^{-1}\bar{u} \tan \varphi), \mathcal{G}, \bar{\theta}_\varphi$ and $\bar{\theta}_p$ on the left of (3.7) may be taken to be time-independent, with error $O(a^4)$.

Let

$$D_t \equiv \partial/\partial t + R^{-1}\bar{u}\partial/\partial\lambda. \quad (3.9)$$

In place of the $O(a)$ particle displacements ξ', η', ζ' of I, we define ξ', η', p^l such that

$$D_t \eta' = v', D_t p^l = \omega', D_t \xi' - (r_0^{-1}\bar{u} \tan \varphi) \eta' = u', \quad (3.10a)$$

where

$$u^l \equiv u' + r_0^{-1}\bar{u}_\varphi \eta' + \bar{u}_p p^l, \quad (3.10b)$$

$$\bar{\xi}' = \bar{\eta}' = \bar{\zeta}' = 0, \quad (3.10c)$$

$$R^{-1}\xi'_\lambda + R^{-1}(\eta' \cos \varphi)_\varphi + p^l_p = 0. \quad (3.10d)$$

Also

$$q' \equiv -\theta' - r_0^{-1}\bar{\theta}_\varphi \eta' - \bar{\theta}_p p^l, \quad (3.11a)$$

so that $\bar{q}' = 0$ and, from the linearized version of (3.1d),

$$D_t q' = Q'. \quad (3.11b)$$

Rather than giving the form corresponding to I(5.9a), we display the slightly more compact form

corresponding to substitution of I(5.6) into I(5.5a):

$$\begin{aligned} & \frac{1}{r_0 \cos^2 \varphi} (S_{(\lambda \varphi)} \cos^2 \varphi)_\varphi + (S_{(\lambda p)})_p \\ &= \frac{1}{r_0 \cos^2 \varphi} (\overline{\eta' X'} \cos^2 \varphi)_\varphi + (\overline{p' X'})_p \\ & \quad + R^{-1} (\overline{\xi_\lambda X'} + \overline{\eta_\lambda Y'} + \overline{\mathcal{G} p' \lambda q'}) \\ & \quad - R^{-1} \mathfrak{B} (\overline{\eta' Q'} \cos \varphi / \overline{\theta}_p)_\varphi + \mathfrak{A} (\overline{\eta' Q'} / \overline{\theta}_p)_p \\ & \quad + \frac{\partial}{\partial t} \left[\frac{1}{r_0 \cos^2 \varphi} (\overline{\eta' u'} \cos^2 \varphi)_\varphi + (\overline{p' u'})_p \right] \\ & \quad + R^{-1} \left\{ \overline{\xi_\lambda (u' - \mathcal{C} \eta')} + \overline{\eta_\lambda v'} \right\} + \frac{1}{r_0 \cos^2 \varphi} \\ & \quad \times \left\{ \left(-\mathfrak{B} \frac{\overline{\eta' \theta'} + \frac{1}{2} r_0^{-1} \overline{\theta}_\varphi \overline{\eta'^2}}{\overline{\theta}_p} + \frac{1}{2} \mathfrak{A} \overline{\eta'^2} \right) \cos^2 \varphi \right\}_\varphi \\ & \quad + \left\{ -\mathfrak{A} \frac{\overline{\eta' q'} + \frac{1}{2} r_0^{-1} \overline{\theta}_\varphi \overline{\eta'^2}}{\overline{\theta}_p} + \frac{1}{2} \mathfrak{B} \overline{p'^2} \right\}_p \Big] + O(a^3). \end{aligned} \quad (3.12a)$$

In place of I(5.9b) we have

$$\begin{aligned} & \frac{\partial}{\partial p} \left(\overline{\omega' \theta'} + \frac{\overline{v' \theta' \overline{\theta}_\varphi}}{r_0 \overline{\theta}_p} \right) \\ &= - \left(\frac{\overline{\theta' Q'}}{\overline{\theta}_p} \right)_p + \frac{\partial}{\partial t} \left[- \left(\frac{\frac{1}{2} \overline{\theta'^2}}{\overline{\theta}_p} \right)_p \right] + O(a^4). \end{aligned} \quad (3.12b)$$

In (3.12a) we have defined

$$\mathcal{C} \equiv r_0^{-1} \overline{u} \tan \varphi + f.$$

For the case of a perfect gas, note from (2.2) and (2.4) that

$$\mathfrak{G} \equiv \frac{\overline{\mathcal{F}}_\theta}{\overline{\mathcal{F}}^2} = \frac{(p_\infty/p)^{1/\gamma}}{\rho_\infty T_\infty} = \frac{\kappa c_p}{p_\infty^* p^{1/\gamma}} \quad (3.13)$$

in the usual notation; in this case \mathfrak{G} happens to be independent of $\overline{\theta}$.

Eqs. (3.12) constitute the *generalized Eliassen-Palm relations* in pressure coordinates. They are consequences of the linearized disturbance equations corresponding to (3.1), and therefore hold for any disturbance which satisfies these equations. When substituted into the transformed mean-flow problem (3.7) they show that all the $O(a^2)$ forcing terms on the right either take the form of time derivatives, or depend explicitly on the non-conservative effects X, Y, Q or q' .

If $X = Y = Q = q' = 0$, we have the *generalized Charney-Drazin theorem* that the $O(a^2)$ forcing of the mean flow (3.8) by conservative waves takes the time-differen-

tiated form $\partial[\overline{\quad}]/\partial t$; here $\overline{\quad}$ stands for the expressions within square brackets on the right of (3.7b), (3.7c), (3.12a) or (3.12b). It then follows that only temporary $O(a^2)$ wave-induced mean-flow changes (see I, footnote 4) can be forced by a transient wave which is itself "temporary" in the sense that all the expressions $\overline{\quad}$ are zero outside some finite time interval; this requires that the meridional displacements η' and p' , in particular, be zero outside the time interval.

In meteorological applications, the disturbance dynamics usually involves restoring effects in the φ and p directions, and we would indeed expect η' and p' to return to zero after a conservative wave has propagated away². It should be kept in mind, however, that "conservative" may in practice be a stringent requirement: boundaries or singular lines might prevent or retard the dispersal of disturbances by conservative processes alone, giving dissipation more time to act than would appear at first sight. This could be the case whether a critical line is acting as an absorber, or as a partial reflector (Béland 1976). Holton and Mass (1976) give a particularly striking model example where small dissipation becomes important because of an approach to critical-line conditions *caused* initially by the effect of wave transience on the mean flow.

4. The virial theorem

If we take the "scalar product" of (ξ', η', p') with the linearized versions of (3.1a,b,c) and use Eqs. (3.11a), (3.10a,b,d) and the thermal wind equation

$$(2\mathcal{C} - f) \overline{u}_p + r_0^{-1} \mathfrak{G} \overline{\theta}_\varphi = O(a^2)$$

we obtain

$$\begin{aligned} & \overline{u'^2} + \overline{v'^2} + (2\mathcal{C} - f) \overline{\xi' v'} \\ & \quad - \overline{f \eta' u'} - \mathfrak{G} \overline{\theta}_p \overline{p'^2} - 2r_0^{-1} \mathfrak{G} \overline{\theta}_\varphi \overline{\eta' p'} \\ & \quad + \{ (2\mathcal{C} - f) r_0^{-1} \overline{u}_p + r_0^{-2} \overline{u}^2 \tan^2 \varphi \} \overline{\eta'^2} \end{aligned}$$

² A proof is easy under the hypothesis that $\nabla \overline{\theta}$ is nowhere parallel to ∇Z , where Z is Ertel's potential vorticity, and that neither gradient vanishes anywhere. For conservative motion we may take $q' = 0$ and hence

$$r_0^{-1} \overline{\theta}_\varphi \eta' + \overline{\theta}_p p' = -\theta',$$

by (3.11a). Similarly [since by Ertel's theorem Z satisfies the same equation (3.1d) as θ , with right-hand side zero],

$$r_0^{-1} \overline{Z}_\varphi \eta' + \overline{Z}_p p' = -Z'.$$

We are here, of course, taking η' and p' to be zero before the onset of the disturbance. Now if u', v', ω' and θ' return to zero afterward, then so does Z' . The foregoing pair of equations for η' and p' then have right-hand sides zero. By hypothesis, the determinant $r_0^{-1} (\overline{\theta}_\varphi \overline{Z}_p - \overline{\theta}_p \overline{Z}_\varphi) \neq 0$, and so η' and p' must also return to zero. A finite-amplitude version of this theorem is given in Andrews and McIntyre (1977). It should be noted, incidentally, that the hypothesis about $\nabla \overline{\theta}$ and ∇Z does not constrain the mean flow to be stable to non-axisymmetric disturbances (cf. Charney and Stern, 1962; Blumen 1968), since it does not exclude the possible presence of boundaries on which $\overline{\theta}$ varies.

$$= -\frac{1}{2} \frac{\partial^2}{\partial t^2} (\overline{\xi'^2 + \eta'^2}) + R^{-1} (\overline{\eta' \Phi'} \cos \varphi)_\varphi + (\overline{p^l \Phi'})_p + (\overline{\xi' X'} + \overline{\eta' Y'} + \mathcal{G} \overline{p^l q'}) + O(a^3). \quad (4.1)$$

This may be called a generalized virial theorem for the disturbance (cf. Eckart, 1963; Hill, 1964). Under the tall-geometry, large-Richardson-number assumptions of I Section 6, it can be shown to reduce to the generalized "equipartition" law I(7.5). (To recover the Boussinesq case, set $\mathcal{G}=1$.) However, Eq. (4.1), and therefore I(7.5), requires no *a priori* assumption concerning the phase relationship between u' and η' , and so we could in fact have obtained I(7.4) directly from I(7.3) without using such a postulate.³

From Eq. (3.13) and Lorenz [(1955), Eq. (8)] it can be seen that the term $-\mathcal{G} \overline{\theta_p p^{l2}}$ in (4.1) reduces to minus twice Lorenz' approximate expression for available potential energy per unit mass, whenever p^l may be approximated by $-\theta'/\overline{\theta_p}$, i.e. whenever the terms q' and $r_0^{-1} \overline{\theta_p \eta'}$ may be neglected to leading order in (3.11a) [or diabatic effects and horizontal advection neglected in the linearized version of (3.1d)]. For plane, sinusoidal, conservative waves in a stratified fluid at rest in a uniform gravitational field [$r_0 \rightarrow \infty$, $f \rightarrow 0$, r.h.s. (4.1) $\rightarrow 0$], Eq. (4.1) reduces to the classical theorem of equipartition of energy, for hydrostatic internal gravity waves.

5. The reduction to Boyd's results

For waves of the special type (1.1) considered by Boyd (1976), with real zonal wavenumber m and complex phase speed

$$\sigma \equiv (\varpi + i\delta)/m,$$

the relation between u' , v' , ω' and the displacements ξ' , η' , p^l become particularly simple. We have

$$(u', v', \omega', \Phi', X', Q', \dots) = \text{Re}\{(\hat{u}^l, \hat{v}^l, \hat{\omega}^l, \hat{\Phi}^l, \hat{X}^l, \hat{Q}^l, \dots) e^{im(\lambda - \sigma t)}\}, \quad (5.1)$$

where $\hat{u}^l = \hat{u}^l(\varphi, p)$, etc. Then

$$D_t = im(R^{-1}\bar{u} - \sigma),$$

and (3.10a) and (3.11b) become

$$im(R^{-1}\bar{u} - \sigma)(\hat{\xi}, \hat{\eta}, \hat{p}^l, \hat{q}) = (\hat{u}^l + (\bar{u}/r_0) \tan \varphi \hat{\eta}, \hat{v}^l, \hat{\omega}^l, \hat{Q}), \quad (5.2)$$

where $\hat{\xi}$, $\hat{\eta}$, \hat{p}^l and \hat{q} are the complex amplitudes of ξ' , η' , p^l and q' . Use of (5.2) to eliminate ξ' , η' , p^l and q'

from the generalized Eliassen-Palm relation (3.12a) immediately produces a result which, like Boyd's (3.9), involves only the familiar fields u' , v' , ω' , etc. Similar corollaries of (3.12a) can easily be written down for any superposition of sinusoidal waves.

However, the result for a single sine wave takes a different form from that of Boyd, except in the case of steady waves (σ real). When σ is complex, more analysis is needed to make the connection. First, it is convenient to eliminate the term $R^{-1}\{\overline{\xi_\lambda'(u' - \mathcal{C}\eta')} + \overline{\eta_\lambda' v'}\}$ from the time-differentiated part of (3.12a) by means of (5.2) and the identity

$$\frac{1}{2} \frac{\partial}{\partial t} (\overline{\xi'^2 + \eta'^2}) = \overline{\xi'(u' - \mathcal{C}\eta')} + \overline{\eta' v'} + (2\mathcal{C} - f) \overline{\xi' \eta'}, \quad (5.3)$$

which follows directly from (3.10a). We also need the relation

$$\hat{\Phi} = -(\bar{u} - R\sigma)(\hat{u}^l - \mathcal{C}\hat{\eta}) - (im)^{-1} R \hat{X}, \quad (5.4)$$

which follows from the linearized zonal momentum equation. Substitution of (5.3) into the first term on the right of the virial theorem (4.1) and use of (5.4) in the second and third terms, followed by lengthy but straightforward calculations [involving the thermal wind equation (3.11a) and specialization to the perfect-gas equation (2.2)] then leads to an equation almost identical to Boyd's (3.9). The only difference lies in the fact that we do not need to make the approximation made by Boyd in the thermodynamic energy equation, namely neglect of the term κT_1 , in his notation. This has the sole result that N^2 in Boyd's Eqs. (VI) and (VII) is replaced (in our version of them) by the exact buoyancy frequency squared, i.e.,

$$N^2 \equiv R(\bar{T}_z + \kappa \bar{T}) \quad [\bar{T} = T_0(z) + T_1], \quad (5.5)$$

again in his (dimensionless) notation.

We have also checked directly that Boyd's analysis for steady, conservative waves [the analysis up to his (2.34)] goes through without the neglect of κT_1 , provided N^2 is first redefined as in (5.5). New terms $-\kappa \overline{\phi_z' w'}$ and $\kappa \overline{\Phi_{z\theta}/a}$ appear on the right of his (2.8) and (2.30) respectively, but the resulting extra terms that arise in his (2.34) can be shown to cancel correct to $O(a^2)$, by virtue of the thermal wind relation and his (2.14). Boyd's (2.32) likewise remains true if N^2 is redefined as in (5.5); and upon transforming the mean-flow problem as in (3.2) above or as in Boyd's Eqs. (VI) and (VII), we may recover the Charney-Drazin theorem for the case of steady waves⁴.

³ An alternative approach is to note that the requirement I(A21) is unnecessarily stringent: a reexamination of Appendix A of I shows that I(7.4) can be derived using the weaker condition $(\overline{p\eta'})_v \approx (\overline{p\eta'})_v$, where τ is "slow" time. But the latter condition is automatically implied by the scaling adopted in I; it follows from the identity $\overline{\xi_{\mathcal{T}} \cdot \mathbf{M}} = \overline{\xi \cdot \mathbf{M}_{\mathcal{T}}}$, where $\mathbf{M} = \mathbf{0}$ is a symbolic representation of the three equations {(4.1a), (4.1b), (5.3b)} of I.

⁴ Note that the "uniqueness" argument at the end of Boyd's Section 2 is incorrect as it stands. This can be seen from the fact that it proves too much; the mean flow can of course be changed by external causes, expressed for instance in boundary conditions. It is only the forcing of the mean flow by the waves which vanishes.

6. The general, nonhydrostatic case

a. Equations

The essential requirement of the theorems is that the mean flow be axisymmetric. In an elementary derivation it is simplest to use cylindrical polar coordinates (R, λ, z) where R is radius, λ is longitude or azimuth, and z is measured parallel to the axis of symmetry. Cylindrical coordinates are directly useful in applications such as hurricane or tornado models; and results for a straight channel may be obtained by setting $R=R_0+x$ and letting $R_0 \rightarrow \infty$. We shall in any case state the final results in a coordinate-independent or vector form, so that they may easily be rewritten in sphericals or in any other axisymmetric coordinates desired.

The geopotential Φ is allowed to be an arbitrary function of meridional position, i.e.,

$$\Phi = \Phi(R, z).$$

This includes the case in which Φ is taken as spherically symmetric as well as the case $\Phi = \Phi(z)$ appropriate to hurricane models. We adopt a rotating frame of reference, with constant angular velocity $(0, 0, \Omega)$, and absorb the centrifugal potential into Φ in the usual way. A nonrotating frame would suffice in principle, since there is no restriction on the magnitude of the mean zonal flow, but we include rotation for familiarity's sake and for convenience in applications. The possibility of a beta effect is included, because the direction of gravity is allowed to vary with R and z .

Let (u, v, w) be the radial, azimuthal and axial velocity components of the velocity \mathbf{u} . We shall take the following as the exact equations, in "flux" form. The azimuthal momentum equation is written first:

$$(\rho v)_t + R^{-2}(R^2 \rho u w)_R + R^{-1}(\rho v^2)_\lambda + (\rho v w)_z + 2\Omega \rho u + R^{-1} p_\lambda = -\rho Y \quad (6.1a)$$

$$(\rho w)_t + R^{-1}(R \rho u w)_R + R^{-1}(\rho v w)_\lambda + (\rho w^2)_z + \rho \Phi_z + p_z = -\rho Z \quad (6.1b)$$

$$(\rho u)_t + R^{-1}(R \rho u^2)_R + R^{-1}(\rho v u)_\lambda + (\rho w u)_z - R^{-1} \rho v^2 - 2\Omega \rho v + \rho \Phi_R + p_R = -\rho X \quad (6.1c)$$

$$(\rho \theta)_t + R^{-1}(R \rho u \theta)_R + R^{-1}(\rho v \theta)_\lambda + (\rho w \theta)_z = -\rho Q \quad (6.1d)$$

$$\rho_t + R^{-1}(R \rho u)_R + R^{-1}(\rho v)_\lambda + (\rho w)_z = 0. \quad (6.1e)$$

The terms on the right represent arbitrary body forces and heating, as before. The equation of state (2.1) completes the set.

b. Zonal-mean problem

Upon taking the zonal average of the flux forms of the equations all the terms involving differentiation with respect to λ vanish. It proves convenient to de-

compose the surviving flux terms in the way typified by

$$\overline{(\rho v w)}_z = \overline{(\rho w \bar{v})}_z + \{(\rho w)' \bar{v}'\}_z.$$

This is because of the way $\bar{\rho u}$ and $\bar{\rho w}$, rather than \bar{u} and \bar{w} , arise naturally in taking the zonal average of (6.1e). As in I, Section 3, we shall treat the total forcing of the mean flow, and hence $\bar{\rho u}$, $\bar{\rho w}$, \bar{u} , \bar{w} , \bar{v}_t , etc., as being $O(a^2)$. Then

$$(\bar{p} \bar{v})_t + R^{-2}(R^2 \bar{\rho} u \bar{v})_R + (\bar{\rho} w \bar{v})_z + 2\Omega \bar{\rho} u = -(\bar{\rho}' \bar{v}')_t - R^{-2}\{R^2(\bar{\rho} u)' \bar{v}'\}_R - \{(\bar{\rho} w)' \bar{v}'\}_z - \bar{\rho} \bar{Y} \quad (6.2a)$$

$$(\bar{\rho} w)_t + \bar{p} \Phi_z + \bar{p}_z = -R^{-1}\{R(\bar{\rho} u)' \bar{w}'\}_R - \{(\bar{\rho} w)' \bar{w}'\}_z - \bar{\rho} \bar{Z} + O(a^4) \quad (6.2b)$$

$$(\bar{\rho} u)_t - R^{-1} \bar{p} \bar{v}^2 - 2\Omega \bar{\rho} \bar{v} + \bar{p} \Phi_R + \bar{p}_R = -R^{-1}[\{R(\bar{\rho} u)' \bar{u}'\}_R - \bar{\rho}'(\bar{v}^2)' - \bar{p} \bar{v}'^2] - \{(\bar{\rho} w)' \bar{u}'\}_z + 2\Omega \bar{\rho}' \bar{v}' - \bar{\rho} \bar{X} + O(a^4) \quad (6.2c)$$

$$(\bar{p} \bar{\theta})_t + R^{-1}(R \bar{\rho} u \bar{\theta})_R + (\bar{\rho} w \bar{\theta})_z = -(\bar{\rho}' \bar{\theta}')_t - R^{-1}\{R(\bar{\rho} u)' \bar{\theta}'\}_R - \{(\bar{\rho} w)' \bar{\theta}'\}_z - \bar{\rho} \bar{Q} \quad (6.2d)$$

$$\bar{p}_t + R^{-1}(R \bar{\rho} u)_R + (\bar{\rho} w)_z = 0. \quad (6.2e)$$

In the next subsection we shall use the fact that, in virtue of (6.2e), the left-hand side of (6.2a) can be written

$$\bar{p} \bar{v}_t + R^{-1} \bar{\rho} \mathbf{u} \cdot \nabla \mathfrak{M}, \quad (6.3)$$

where

$$\mathfrak{M}(R, z, t) = R \bar{v} + R^2 \Omega, \quad (6.4)$$

the mean specific angular momentum about the symmetry axis. Similarly, the left-hand side of (6.2d) is equal to

$$\bar{p} \bar{\theta}_t + \bar{\rho} \mathbf{u} \cdot \nabla \bar{\theta}, \quad (6.5)$$

since $\bar{\theta} = \bar{\theta}(R, z, t)$.

By eliminating Φ between (6.2b,c) we find that the relevant form of the thermal-wind equation is

$$2\bar{p} R^{-2} \mathfrak{M} \bar{v}_z + \bar{p}^{-1}(\bar{p}_z \bar{p}_R - \bar{p}_R \bar{p}_z) = O(a^2). \quad (6.6)$$

c. A coordinate-independent preliminary transformation.

The preliminary transformations of the mean-flow problem introduced in I and in (3.2) above subtract out the part of the mean meridional circulation which is proportional to the horizontal eddy flux of potential temperature. Thus they treat the horizontal direction as special, which is convenient but somewhat arbitrary. The same can easily be done here; but from a fundamental viewpoint it seems more natural to use the local eddy-flux component *along the mean isentropes*. We make this choice here, and define a mass stream-

function ψ proportional to the vector product of the meridional eddy flux with $\nabla\bar{\theta}$:

$$R^{-1}\psi(R,z) \equiv \{(\overline{\rho w})'\bar{\theta}'_R - \overline{(\rho u)'\theta}'_z\} / |\nabla\bar{\theta}|^2 \quad (6.7a)$$

$$\equiv \overline{(\rho \mathbf{u})'\theta}' \otimes \nabla\bar{\theta} / |\nabla\bar{\theta}|^2, \quad (6.7b)$$

say⁵. The shorthand notation \otimes is defined to mean the azimuthal component of the full vector product. Then the remaining contribution $\mathbf{U}^* = (U^*, 0, W^*)$ to the mean meridional mass flux is defined by

$$\overline{\rho u} = U^* - R^{-1}\psi_z, \quad \overline{\rho w} = W^* + R^{-1}\psi_R. \quad (6.8)$$

[Alternatives to (6.7) which are formally closer to (3.2) would be $\psi = \overline{(\rho w)'\theta}'/\bar{\theta}_R$ or $-\overline{(\rho u)'\theta}'/\bar{\theta}_z$; it can be shown that either would lead to *coordinate-dependent* generalized Eliassen-Palm relations.] The quantities

$$\left. \begin{aligned} S_{(\lambda R)} &= \overline{(\rho u)'\theta}' + R^{-2}\mathfrak{M}_z\psi \\ S_{(\lambda z)} &= \overline{(\rho w)'\theta}' - R^{-2}\mathfrak{M}_R\psi \end{aligned} \right\}$$

will now appear in the transformed equation for \bar{v}_t , in the combination

$$R^{-2}(R^2S_{(\lambda R)})_R + (S_{(\lambda z)})_z, \quad (6.9)$$

which we shall denote for brevity by

$$R^{-1}\nabla\odot(RS_{(\lambda)}).$$

Here $\nabla\odot\mathbf{F}$ is shorthand for the divergence of the projection of any vector \mathbf{F} onto the meridional plane. The meridional components $S_{(\lambda R)}$ and $S_{(\lambda z)}$ of $\mathbf{S}_{(\lambda)}$ are related [by Eqs. (A8)] to the meridional fluxes of angular momentum in a generalized Lagrangian-mean description. [This is further explained in McIntyre (1977).]

The ψ contribution to the last term in (6.5) may be combined with the eddy flux divergence on the right of (6.2d) to give (noting $|\nabla\bar{\theta}|^2 = \bar{\theta}_R^2 + \bar{\theta}_z^2$):

$$\begin{aligned} R^{-1}(\psi_R\bar{\theta}_z - \psi_z\bar{\theta}_R) + \{(\overline{\rho w})'\theta}'_z + R^{-1}\{R(\overline{\rho u})'\theta}'_R \\ = R^{-1}\{\psi\bar{\theta}_z + R(\overline{\rho u})'\theta}'_R + R^{-1}\{-\psi\bar{\theta}_R + R(\overline{\rho w})'\theta}'_z \\ = R^{-1}(R\bar{\theta}_R\chi)_R + (\bar{\theta}_z\chi)_z = \nabla\odot(\chi\nabla\bar{\theta}) \end{aligned} \quad (6.10)$$

in virtue of (6.7a), where

$$\begin{aligned} \chi &= \{(\overline{\rho u})'\theta}'_R + \overline{(\rho w)'\theta}'_z\} / |\nabla\bar{\theta}|^2 \\ &= \overline{(\rho \mathbf{u})'\theta}' \cdot \nabla\bar{\theta} / |\nabla\bar{\theta}|^2. \end{aligned} \quad (6.11)$$

⁵ This definition appears to assume that the stratification $\nabla\bar{\theta} \neq 0$. It can however be shown that ψ remains finite in the limit of vanishing stratification, provided Q' vanishes at least as fast as $|\nabla\bar{\theta}|$. In that limit, we may re-express $R^{-1}\psi$ as $\bar{\rho}'u' +$ (terms of time-differentiated form) + (terms involving $Q'/|\nabla\bar{\theta}|$); see Eqs. (6.19) and (A5a).

Using (6.3)–(6.10) we may now rewrite (6.2a,d) as

$$\bar{p}\bar{v}_t + \mathbf{U}^* \cdot \nabla\mathfrak{M} = -R^{-1}\nabla\odot(RS_{(\lambda)}) - \overline{(\rho'v)'}_t - \bar{\rho}Y \quad (6.12a)$$

$$\bar{p}\bar{\theta}_t + \mathbf{U}^* \cdot \nabla\bar{\theta} = -\nabla\odot(\chi\nabla\bar{\theta}) - \overline{(\rho'\theta)'}_t - \bar{\rho}Q, \quad (6.12d)$$

where $\mathbf{U}^* = (U^*, 0, W^*)$. After substitution from (6.8) and differentiation with respect to time, (6.2b,c) take the form

$$W^*_t + \mathfrak{F}_z\bar{p}_t + \bar{p}_{zt} = -\frac{\partial}{\partial t}[\text{RHS of (6.2b)} - R^{-1}\psi_{Rt}], \quad (6.12b)$$

$$\begin{aligned} U^*_t + \mathfrak{M}\bar{p}_t - 2\bar{p}R^{-2}\mathfrak{M}\bar{w}_t + \bar{p}_{Rt} \\ = -\frac{\partial}{\partial t}[\text{RHS of (6.2c)} + R^{-1}\psi_{zt}], \end{aligned} \quad (6.12c)$$

where $\mathfrak{M}(R,z,t) \equiv \mathfrak{F}_R - R^{-1}\bar{v}^2 - 2\Omega\bar{v}$. The transformed continuity and time-differentiated state equations are

$$\bar{p}_t + \nabla\odot\mathbf{U}^* = 0, \quad (6.12e)$$

$$\bar{p}_t - \mathfrak{F}_z\bar{\theta}_t - \mathfrak{F}_z\bar{p}_t = -\frac{\partial}{\partial t}[\text{RHS of (2.6)}]. \quad (6.12f)$$

Eqs. (6.12) comprise a complete set of equations for the $O(a^2)$ mean-flow quantities

$$\{\bar{v}_t, \bar{\theta}_t, \bar{p}_t, \bar{p}_t, W^*, U^*\}. \quad (6.13)$$

The coefficients \bar{p} , \mathfrak{M} , $\bar{\theta}_z$, $\bar{\theta}_R$, \mathfrak{M} , \mathfrak{F}_z and \mathfrak{F}_z may be taken as time-independent, i.e., as functions of R and z alone, with error $O(a^4)$.

d. The generalized Eliassen-Palm relations

As before, these are consequences of the linearized disturbance equations, and will show that, correct to $O(a^2)$, the expressions

$$-R^{-1}\nabla\odot(RS_{(\lambda)}) \quad \text{and} \quad -\nabla\odot(\chi\nabla\bar{\theta}) \quad (6.14)$$

on the RHS of (6.12a) and (6.12d), respectively, are equal to sums of time-differentiated terms and terms explicitly involving the forcing or dissipation of the waves by $\mathbf{X}' = (X', Y', Z')$, Q' and q' .

It facilitates the analysis if we linearize Eqs. (6.1) after expressing (6.1a–d) in “material” rather than “flux” form, that is, after dividing by ρ and using the continuity equation (6.1e) to eliminate ρ_t . With the definition

$$D_t \equiv \partial/\partial t + R^{-1}\bar{v}\partial/\partial\lambda,$$

the resulting equations, correct to $O(a)$, are

$$D_tv' + R^{-1}\mathbf{u}' \cdot \nabla\mathfrak{M} + (R\bar{p})^{-1}p'_\lambda = -Y', \quad (6.15a)$$

$$D_tv' - \bar{p}_z\bar{p}^{-2}p' + \bar{p}^{-1}p'_z = -Z', \quad (6.15b)$$

$$D_tv' - 2R^{-2}\mathfrak{M}v' - \bar{p}_R\bar{p}^{-2}p' + \bar{p}^{-1}p'_R = -X', \quad (6.15c)$$

$$D_t \theta' + \mathbf{u}' \cdot \nabla \bar{\theta} = -Q', \quad (6.15d)$$

$$D_t \rho' + \mathbf{u}' \cdot \nabla \bar{\rho} + \bar{\rho} \nabla \cdot \mathbf{u}' = 0, \quad (6.15e)$$

$$\rho' - \bar{\rho} \theta' - \bar{\rho} p' = 0, \quad (6.15f)$$

the last being just Eq. (2.5), the linearized equation of state.

We now introduce a vector $\xi' = (\xi', \eta', \zeta')$, the $O(a)$ particle displacement associated with the wave motion. It is defined to satisfy

$$(\partial/\partial t + \bar{\mathbf{u}} \cdot \nabla) \xi' = \mathbf{u}' \quad (6.16)$$

correct to $O(a)$, where \mathbf{u}' is the $O(a)$ part of

$$\mathbf{u}' + \xi' \cdot \nabla \bar{\mathbf{u}}, \quad (6.17)$$

i.e., the $O(a)$ approximation to the Lagrangian disturbance velocity defined in Andrews and McIntyre (1977). Noting that $\bar{\mathbf{u}} = (0, \bar{v}, 0) + O(a^2)$, we have

$$\mathbf{u}' = (\mathbf{u}' - R^{-1} \bar{v} \eta', v' + \bar{v}_R \xi' + \bar{v}_z \zeta', w') \quad (6.18)$$

$$= (u', v', w'),$$

say; and⁶

$$D_t \xi' = w', \quad D_t \eta' = u', \quad D_t \zeta' + R^{-1} \bar{v} \xi' = v'. \quad (6.19a)$$

We require also

$$\bar{\xi}' = \bar{\eta}' = \bar{\zeta}' = 0, \quad (6.19b)$$

and

$$\nabla \cdot \xi' \equiv R^{-1} (R \xi')_R + R^{-1} \eta'_\lambda + \zeta'_z = -(\rho'/\bar{\rho}) + O(a^3), \quad (6.19c)$$

where

$$\rho' \equiv \rho' + \xi' \cdot \nabla \bar{\rho} \equiv \rho' + \bar{\rho}_R \xi' + \bar{\rho}_z \zeta', \quad (6.20)$$

the Lagrangian disturbance density. Note that (6.19a) implies $D_t \bar{\xi}' = D_t \bar{\eta}' = D_t \bar{\zeta}' = 0$, so that the requirement (6.19b) is consistent with (6.19a). Similarly, taking the divergence of (6.16) or (6.19a) we get $D_t (\nabla \cdot \xi') = \nabla \cdot \mathbf{u}'$, which is consistent with (6.19c) since (6.15e) may be rewritten as

$$D_t \rho' + \bar{\rho} \nabla \cdot \mathbf{u}' = 0, \quad (6.21)$$

and since $\bar{\rho}_t = O(a^2)$. We also define

$$q' \equiv -\theta' - \xi' \cdot \nabla \bar{\theta} \equiv -\theta' - \bar{\theta}_R \xi' - \bar{\theta}_z \zeta', \quad (6.22a)$$

so that $\bar{q}' = 0$ and, from (6.15d) and (6.19a),

$$D_t q' = Q'. \quad (6.22b)$$

Given the foregoing definitions, and the foregoing choices for the forms of the disturbance equations (6.15)

⁶ The cylindrical components of \mathbf{u}' are resolved onto the coordinate directions at the point (R, λ, z) , not at the displaced position $(R + \xi', \lambda + R^{-1} \eta', z + \zeta')$. In the second of (6.19a), the metric term $-R^{-1} \bar{v} \eta'$ does not appear on the right because it is cancelled by an equal term arising on the left of (6.16). The metric term in the third of (6.19a) is analogous to that in (3.10a); but note that (3.10a) has no contribution involving the vertical displacement, as a result of the "thin-shell" type of approximation used there.

and transformed mean-flow problem (6.12), the remaining analysis becomes a moderately straightforward extension of that in I. First, it is shown in the Appendix that

$$\begin{aligned} & R^{-1} \nabla \odot (R S_{(\lambda)}) \\ &= \frac{1}{R} \nabla \odot (\bar{\rho} R \xi' Y') + \frac{\bar{\rho}}{R} (\xi'_\lambda \cdot \mathbf{X}' + \mathcal{G} \rho'_\lambda q') \\ &\quad - R^{-2} \nabla \otimes \left\{ \frac{\bar{\rho} Q' \xi' \otimes \nabla \bar{\theta}}{|\nabla \bar{\theta}|^2} R \nabla \mathcal{M} \right\} \\ &\quad + \frac{\partial}{\partial t} \left[\frac{1}{R} \nabla \odot (\bar{\rho} R \xi' v') + \frac{\bar{\rho}}{R} (\xi'_\lambda \cdot \mathbf{u}' + \xi'_\lambda \cdot \boldsymbol{\Omega} \times \xi') \right] \\ &\quad - R^{-2} \nabla \otimes \left\{ \frac{\bar{\rho} (\theta' + \frac{1}{2} \xi' \cdot \nabla \bar{\theta}) \xi' \otimes \nabla \bar{\theta}}{|\nabla \bar{\theta}|^2} R \nabla \mathcal{M} \right\} \\ &\quad + \frac{1}{2R} \nabla \odot \{ \bar{\rho} \xi' (\xi' \cdot \nabla \mathcal{M}) \} + O(a^3), \quad (6.23a) \end{aligned}$$

where ξ'_λ is defined to mean $(\xi'_\lambda, \eta'_\lambda, \zeta'_\lambda)$. As before, $\nabla \odot$ denotes the divergence of the meridional projection of a vector, and \otimes the "meridional" vector product, i.e., the azimuthal component of the full vector product. [Eq. (6.23a) is written out in component form in (A20). Note that by (6.22a) we can replace $\theta' + \frac{1}{2} \xi' \cdot \nabla \bar{\theta}$ in the last line by the alternative expression $-q' - \frac{1}{2} \xi' \cdot \nabla \bar{\theta}$.] Second,

$$\begin{aligned} \nabla \odot (\chi \nabla \bar{\theta}) &= -\nabla \odot \left\{ \frac{\bar{\rho} \rho' Q' \nabla \bar{\theta}}{|\nabla \bar{\theta}|^2} \right\} \\ &\quad - \frac{\partial}{\partial t} \left[\nabla \odot \left\{ \frac{\bar{\rho} \theta'^2 \nabla \bar{\theta}}{2 |\nabla \bar{\theta}|^2} \right\} \right] + O(a^3). \quad (6.23b) \end{aligned}$$

This follows from the fact that

$$\chi |\nabla \bar{\theta}|^2 = -\bar{\rho} \theta' Q' - \frac{1}{2} \partial (\bar{\rho} \theta'^2) / \partial t \quad (6.24)$$

which is the result of multiplying (6.15d) by $\bar{\rho} \theta'$ and averaging, noting that \bar{u} and \bar{w} are $O(a^2)$ so that

$$\bar{\rho} \mathbf{u}' \theta' \cdot \nabla \bar{\theta} = (\bar{\rho} \mathbf{u}') \theta' \cdot \nabla \bar{\theta} + O(a^3)$$

and that $\bar{\rho}_t$ and $\bar{\theta}_t$ are $O(a^2)$.

The results (6.23a,b) constitute the generalized Eliassen-Palm relations. They show that all the discussion in Section 3 and in I, Section 5, applies word for word to the present, far more general problem. In addition, we note the following four points.

1) In I, the Boussinesq approximation especially emphasized the direction of gravity in the equations; here we have been able to obtain results having a more symmetrical form which emphasizes no particular direc-

tion in the meridional plane. If gravity does in fact dominate ∇p , its direction is felt through p^l , which is then dominated by vertical displacements through the hydrostatic pressure gradient. It can thus be seen why a vertical displacement ζ' naturally appears in the Boussinesq version in place of the present p^l , as in the term in $\overline{\zeta'_\lambda q'}$ in I Eq. (5.9a), which corresponds to the terms in $\overline{p^l q'}$ in (3.12a) and (6.23a) above. (In a pressure-coordinate description, p^l is of course the exact analogue of ζ' , as was noted in Section 3.)

2) To write the results in spherical coordinates (λ, φ, r) where λ is longitude and φ latitude as in Section 3, note that for any vector $\mathbf{V} \equiv (V_{(\lambda)}, V_{(\varphi)}, V_{(r)})$

$$\begin{aligned} \nabla \odot \mathbf{V} &= R^{-1} \partial(\cos \varphi V_{(\varphi)}) / \partial \varphi + r^{-2} \partial(r^2 V_{(r)}) / \partial r, \\ \nabla \otimes \mathbf{V} &= r^{-1} \partial V_{(r)} / \partial \varphi - r^{-1} \partial(r V_{(\varphi)}) / \partial r, \end{aligned}$$

where

$$R = r \cos \varphi.$$

3) The explicit appearance of R in (6.23a) reflects the fact that R represents a coordinate-independent physical quantity (the distance to the symmetry axis) as well as serving as a cylindrical coordinate. The appearance of R in a coordinate-independent form of the results is to be expected in view of the relation to Kelvin's circulation theorem mentioned in Section 1, and to the angular momentum principle (Bretherton, 1977).

4) The quantity $\overline{p(\xi'_\lambda \cdot \mathbf{u}^l + \xi'_\lambda \cdot \boldsymbol{\Omega} \times \xi')}$ appearing in the transient part of (6.23a) [see also (A.17) and (A.19)] is minus the density of angular pseudomomentum, a quantity related to the classical "energy-momentum tensor" (Landau and Lifshitz, 1975), and therefore related (almost as closely) to the exact "wave-action" concept proposed by Hayes (1970). The mathematical analysis showing this has been given by Bretherton (1977), and its interpretation further explained in Andrews and McIntyre (1977), McIntyre (1977), and references. A virial theorem analogous to that of Section 4 may easily be derived, by taking the scalar product of ξ with Eqs. (6.15a-c), and is important in making the connection with the approximate wave-action concept of Bretherton and Garrett (1968). However, we again emphasize that the derivations in I and in the present paper do not presume any familiarity with these ideas.

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APPENDIX

Derivation of Eq. (6.23a)

Recalling the identities $\overline{\zeta' D_t v'} = -\overline{v' D_t \zeta'} + \overline{(\zeta' v')_t}$, etc., we multiply (6.15a) by $-\zeta'$ and then by $-\xi'$ [Brether-

ton, 1969, Eq. (18) and Fig. 1] to obtain analogues of I Eqs. (A11):

$$\overline{w' v'} + (R\bar{p})^{-1} \overline{\zeta'_\lambda p'} - \mathcal{Q} \overline{\zeta' u'} = \overline{\zeta' Y'} + \frac{1}{2} \mathcal{B} (\overline{\zeta'^2})_t + \overline{(\zeta' v')_t}, \quad (\text{A1a})$$

$$\begin{aligned} \overline{u' v'} + (R\bar{p})^{-1} \overline{\xi'_\lambda p'} - \mathcal{B} \overline{\xi' w'} \\ = \overline{\xi' Y'} + \frac{1}{2} \mathcal{Q} (\overline{\xi'^2})_t + \overline{(\xi' v')_t}, \quad (\text{A1b}) \end{aligned}$$

where $\mathcal{Q} = R^{-1} \mathcal{N}_R$ and $\mathcal{B} = R^{-1} \mathcal{N}_z$.

The next step is to relate the terms $-\mathcal{Q} \overline{\zeta' u'}$ and $-\mathcal{B} \overline{\xi' w'}$ to the eddy flux terms in the definition (6.7) of ψ . Note first that

$$(\rho w)' = \bar{p} w' + \rho' \bar{w} + \rho' w' - \overline{\rho' w'} = \bar{p} w' + O(a^2),$$

and similarly for $(\rho u)'$, whence (6.7) implies that

$$(R\bar{p})^{-1} |\nabla \bar{\theta}|^2 \psi = \overline{w' \theta'} \bar{\theta}_R - \overline{u' \theta'} \bar{\theta}_z + O(a^3). \quad (\text{A2})$$

Similarly

$$S_{(\lambda R)} = \overline{p u' v'} + R^{-1} \mathcal{B} \psi + O(a^3), \quad (\text{A3a})$$

$$S_{(\lambda z)} = \overline{p w' v'} - R^{-1} \mathcal{Q} \psi + O(a^3). \quad (\text{A3b})$$

The needed relations between $\overline{\zeta' u'}$, $\overline{\xi' w'}$ and $\overline{u' \theta'}$ can be obtained by forming the analogues of (A1) based on (6.15d); alternatively, we may proceed directly from (6.22a). From here on we drop the primes from the symbols representing disturbance fields. Noting that $-\overline{q D_t \xi} = \overline{\xi D_t q} - (\overline{\xi q})_t = \overline{\xi Q} - (\overline{\xi q})_t$, by (6.22b), we multiply (6.22a) by $u = D_t \xi$ to obtain

$$\overline{u \bar{\theta}} = -\bar{\theta}_z \overline{u \zeta} + \overline{\xi Q} - (\overline{\xi q})_t - \frac{1}{2} \bar{\theta}_R (\overline{\xi^2})_t. \quad (\text{A4a})$$

Similarly, multiplying by $w = D_t \zeta$,

$$\overline{w \bar{\theta}} = -\bar{\theta}_R \overline{w \xi} + \overline{\zeta Q} - (\overline{\zeta q})_t - \frac{1}{2} \bar{\theta}_z (\overline{\zeta^2})_t. \quad (\text{A4b})$$

Upon substitution of (A4) into (A2), noting that $\overline{w \xi} = -\overline{u \zeta} + (\overline{\zeta \xi})_t$, we obtain the two alternative forms

$$\begin{aligned} (R\bar{p})^{-1} |\nabla \bar{\theta}|^2 \psi \\ = |\nabla \bar{\theta}|^2 \overline{\zeta u} - \bar{\theta}_R \overline{\zeta \xi}_t + H + \frac{\partial J}{\partial t} + O(a^3) \quad (\text{A5a}) \end{aligned}$$

$$= -|\nabla \bar{\theta}|^2 \overline{\xi w} + \bar{\theta}_z \overline{\zeta \xi}_t + H + \frac{\partial J}{\partial t} + O(a^3), \quad (\text{A5b})$$

where

$$H = \bar{\theta}_R \overline{\zeta Q} - \bar{\theta}_z \overline{\xi Q} = \overline{Q \xi} \otimes \nabla \bar{\theta}, \quad (\text{A6})$$

$$J = -\bar{\theta}_R \overline{\zeta q} + \bar{\theta}_z \overline{\xi q} + \frac{1}{2} \bar{\theta}_R \bar{\theta}_z (\overline{\xi^2} - \overline{\zeta^2}).$$

We now multiply (A5a,b), respectively, by $-\mathcal{Q}/|\nabla \bar{\theta}|^2$ and $\mathcal{B}/|\nabla \bar{\theta}|^2$ and add the results to (A1a,b), noting

that $-\bar{\theta}_R^2 \bar{\zeta} \bar{\xi} + J = |\nabla \bar{\theta}|^2 (-K - \frac{1}{2} \bar{\zeta} \bar{\xi})$, and $\bar{\theta}_z^2 \bar{\zeta} \bar{\xi} + J = |\nabla \bar{\theta}|^2 (-K + \frac{1}{2} \bar{\zeta} \bar{\xi})$ where

$$K = \overline{(q + \frac{1}{2} \bar{\xi} \cdot \nabla \bar{\theta}) \bar{\xi} \otimes \nabla \bar{\theta} / |\nabla \bar{\theta}|^2}. \quad (\text{A7})$$

The results, with all wave dissipation forcing and transience terms written on the right, are

$$\begin{aligned} \bar{p}^{-1} (S_{(\lambda R)} + R^{-1} \bar{\xi}_\lambda \bar{p}) \\ = \bar{\xi} \bar{Y} + \frac{\alpha H}{|\nabla \bar{\theta}|^2} + \frac{\partial}{\partial t} [\alpha (-K + \frac{1}{2} \bar{\zeta} \bar{\xi}) + \frac{1}{2} \alpha \bar{\xi}^2 + \bar{\xi} v] \\ + O(a^3), \quad (\text{A8a}) \end{aligned}$$

$$\begin{aligned} \bar{p}^{-1} (S_{(\lambda z)} + R^{-1} \bar{\zeta}_\lambda \bar{p}) \\ = \bar{\zeta} \bar{Y} - \frac{\alpha H}{|\nabla \bar{\theta}|^2} + \frac{\partial}{\partial t} [\alpha (K + \frac{1}{2} \bar{\zeta} \bar{\xi}) + \frac{1}{2} \alpha \bar{\zeta}^2 + \bar{\zeta} v] \\ + O(a^3). \quad (\text{A8b}) \end{aligned}$$

Here the terms in \bar{p} , namely $\bar{\xi}_\lambda \bar{p}$ and $\bar{\zeta}_\lambda \bar{p}$, are proportional to the Lagrangian-mean meridional fluxes of angular momentum [see Bretherton 1977].

The last step amounts to noting that $-\bar{\xi}_\lambda \bar{p}$ and $-\bar{\zeta}_\lambda \bar{p}$ also represent fluxes of angular pseudomomentum (Peierls, 1976; Andrews and McIntyre, 1977; McIntyre, 1977), and therefore that their divergence,

$$-R^{-1} (R \bar{\xi}_\lambda \bar{p})_R - (\bar{\zeta}_\lambda \bar{p})_z, \quad (\text{A9})$$

equals minus the local rate of change of angular pseudomomentum (another wave-transience term) plus contributions in \mathbf{X} or q representing wave dissipation or forcing. The expression (A9) is just $-R$ times the expression that will arise from the \bar{p} terms when (A8) is substituted into (6.9); the end result will be (6.23a). We now demonstrate this from first principles.

In elementary terms, the required operation is just that suggested by the form of (A9), namely, scalar multiplication of (6.15c,a,b) by $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ [cf. I Eq. (A15); McIntyre, 1977, Eqs. (3.6), (3.11)]. A few more manipulations than in I are necessary, because $\nabla \cdot \bar{\xi} \neq 0$ in a compressible fluid. The end result is comparably simple, however.

To deal with the latter manipulations first, note that since $\partial(\bar{\quad})/\partial \lambda = 0$, (A9) equals

$$-\bar{p} \frac{\partial}{\partial \lambda} \nabla \cdot \bar{\xi} - \bar{\xi}_\lambda \cdot \nabla \bar{p},$$

which by (6.19c) equals

$$\bar{p}^{-1} \bar{p} \rho_\lambda^i - \bar{\xi}_\lambda \cdot \nabla \bar{p} + O(a^4). \quad (\text{A10})$$

The Lagrangian disturbance pressure

$$\bar{p}^i \equiv \bar{p} + \bar{p}_R \bar{\xi} + \bar{p}_z \bar{\zeta}, \quad (\text{A11})$$

the Lagrangian disturbance density ρ^i , and the thermal forcing term $-q$ [which by (6.22a) equals the Lagrangian disturbance potential temperature θ^i] evidently satisfy a linearized equation of state of the same form as (6.15f):

$$\rho^i + \mathcal{F} \bar{p} q - \mathcal{F} \bar{p} \bar{p}^i = O(a^2). \quad (\text{A12})$$

Alternatively, (A12) can be verified directly from (6.15f), (6.20), (6.22a) and (A11), and the R and z derivatives of (2.5). Now by (A11)

$$\bar{p} \rho_\lambda^i = -\bar{p}_\lambda \bar{p}^i = -\bar{p}_\lambda^i \bar{p}^i + \bar{p}_R \bar{\xi}_\lambda \bar{p}^i + \bar{p}_z \bar{\zeta}_\lambda \bar{p}^i.$$

We use (A12) in the first term on the right, and (6.20) in the other two. Since $\bar{\xi}_\lambda \bar{\xi} = \bar{\zeta}_\lambda \bar{\zeta} = 0$ and

$$\mathcal{F} \bar{p} \bar{p}_\lambda^i \bar{p}^i = 0, \quad (\text{A13})$$

we get

$$\begin{aligned} \bar{p} \rho_\lambda^i &= \mathcal{F} \bar{p} \bar{p}_\lambda^i q \\ &+ \bar{p}_R \bar{\xi}_\lambda \bar{p}^i + \bar{p}_z \bar{\zeta}_\lambda \bar{p}^i + (\bar{p}_z \bar{p}_R - \bar{p}_R \bar{p}_z) \bar{\xi}_\lambda \bar{\zeta} + O(a^3). \quad (\text{A14}) \end{aligned}$$

The second term in (A10) is now evaluated by taking the scalar product of $\bar{\xi}_\lambda$ with Eqs. (6.15c,a,b). We note first from (6.19a) that $\bar{\xi}_\lambda D_i u = (\bar{\xi}_\lambda u)_i$, $\bar{\zeta}_\lambda D_i v = (\bar{\zeta}_\lambda v)_i$, and that

$$\begin{aligned} -\bar{\xi}_\lambda 2R^{-2} \mathcal{N} v + \eta_\lambda (D_i v + R^{-1} \mathbf{u} \cdot \nabla \mathcal{N}) \\ = -2R^{-2} \mathcal{N} \bar{\xi}_\lambda (v^i - \bar{v}_R \bar{\xi}^i - \bar{v}_z \bar{\zeta}^i) \\ + \eta_\lambda \{ D_i v^i + (2\Omega + R^{-1} \bar{v}) \eta_\lambda \bar{u} \} \\ = 2R^{-2} \mathcal{N} \bar{v}_z \bar{\xi}_\lambda \bar{\zeta}^i + (\eta_\lambda v^i)_i + (2\Omega + R^{-1} \bar{v}) (\eta_\lambda \bar{\xi})_i, \quad (\text{A15}) \end{aligned}$$

after a little manipulation. Thus

$$\begin{aligned} -\bar{\xi}_\lambda \cdot \nabla \bar{p} &= -\partial P / \partial t + \bar{p} (\bar{\xi}_\lambda X + \eta_\lambda Y + \bar{\zeta}_\lambda Z) \\ -\bar{p}^{-1} (\bar{p}_R \bar{\xi}_\lambda \bar{p} + \bar{p}_z \bar{\zeta}_\lambda \bar{p}) &+ 2\bar{p} R^{-2} \mathcal{N} \bar{v}_z \bar{\xi}_\lambda \bar{\zeta} + O(a^4), \quad (\text{A16}) \end{aligned}$$

where

$$\begin{aligned} P &= -\bar{p} \{ \bar{\xi}_\lambda u + \eta_\lambda v^i + \bar{\zeta}_\lambda w + (2\Omega + R^{-1} \bar{v}) \eta_\lambda \bar{\xi} \} \\ &= -\bar{p} \{ \bar{\xi}_\lambda \cdot \mathbf{u}^i + \Omega (\eta_\lambda \bar{\xi} - \bar{\xi}_\lambda \bar{\eta}) \}, \quad (\text{A17}) \end{aligned}$$

in virtue of (6.18). Now the thermal-wind equation (6.6) is

$$2\bar{p} R^{-2} \mathcal{N} \bar{v}_z + \bar{p}^{-1} (\bar{p}_z \bar{p}_R - \bar{p}_R \bar{p}_z) = O(a^2). \quad (\text{A18})$$

So when, in order to form the expression (A10), Eq. (A16) is added to \bar{p}^{-1} times (A14), the terms on the last line of each cancel, with error $O(a^4)$. The result is

$$\begin{aligned} -R^{-1} (R \bar{\xi}_\lambda \bar{p})_R - (\bar{\zeta}_\lambda \bar{p})_z \\ = -\partial P / \partial t + \bar{p} \bar{\xi}_\lambda \cdot \mathbf{X} + \bar{p}^{-1} \mathcal{F} \bar{p} \bar{p}_\lambda^i q + O(a^3). \quad (\text{A19}) \end{aligned}$$

This equation shows that P is a conservable quantity, being conserved with flux $(-\xi_\lambda \bar{p}, 0, -\zeta_\lambda \bar{p})$ if $X=Y=Z=q=0$; P is in fact the density of angular pseudo-momentum, correct to $O(a^2)$.

Adding R^{-1} times (A19) to

$$R^{-2} \partial \bar{p} R^2 (A8a) / \partial R + \partial \bar{p} (A8b) / \partial z$$

now gives

$$\begin{aligned} & R^{-2} (R^2 S_{(\lambda R)})_R + (S_{(\lambda z)})_z \\ &= R^{-2} (\bar{p} R^2 \xi \bar{Y})_R + (\bar{p} \zeta \bar{Y})_z \\ &+ (\bar{p}/R) (\xi_\lambda \bar{X} + \eta_\lambda \bar{Y} + \zeta_\lambda \bar{Z} + \bar{p}^{-2} \mathfrak{T} \bar{p}'_\lambda q) \\ &+ R^{-2} \{ \bar{p} H R^2 \mathfrak{R} / |\nabla \bar{\theta}|^2 \}_R - \{ \bar{p} H \mathfrak{G} / |\nabla \bar{\theta}|^2 \}_z \\ &+ (\partial/\partial t) [R^{-2} (\bar{p} R^2 \xi \bar{v})_R + (\bar{p} \zeta \bar{v})_z - R^{-1} P \\ &+ R^{-2} \{ \bar{p} R^2 (-\mathfrak{R} K + \frac{1}{2} \mathfrak{G} \xi^2 + \frac{1}{2} \mathfrak{R} \xi \zeta') \}_R \\ &+ \{ \bar{p} (\mathfrak{G} K + \frac{1}{2} \mathfrak{G} \xi \zeta' + \frac{1}{2} \mathfrak{R} \zeta'^2) \}_z] + O(a^3), \quad (A20) \end{aligned}$$

where H , K and P are given by (A6), (A7) and (A17). This is the form of (6.23a) in which $-\theta - \frac{1}{2} \xi \cdot \nabla \bar{\theta}$ is replaced by $q + \frac{1}{2} \xi \cdot \nabla \bar{\theta}$ using (6.22a).

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